

BOOLEAN METHODS IN THE THEORY OF VECTOR LATTICES

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ABSTRACT. This is an overview of the recent results of interaction of Boolean valued analysis and vector lattice theory.

INTRODUCTION

Boolean valued analysis is a general mathematical method that rests on a special model-theoretic technique. This technique consists primarily in comparison between the representations of arbitrary mathematical objects and theorems in two different set-theoretic models whose constructions start with principally distinct Boolean algebras. We usually take as these models the cosiest Cantorian paradise, the von Neumann universe of Zermelo–Fraenkel set theory, and a special universe of Boolean valued “variable” sets trimmed and chosen so that the traditional concepts and facts of mathematics acquire completely unexpected and bizarre interpretations. The use of two models, one of which is *formally* nonstandard, is a family feature of *nonstandard analysis*. For this reason, Boolean valued analysis means an instance of nonstandard analysis in common parlance. By the way, the term *Boolean valued analysis* was minted by G. Takeuti.

Proliferation of Boolean valued models is due to P. Cohen’s final breakthrough in Hilbert’s Problem Number One. His method of forcing was rather intricate and the inevitable attempts at simplification gave rise to the Boolean valued models by D. Scott, R. Solovay, and P. Vopěnka.

Our starting point is a brief description of the best Cantorian paradise in shape of the von Neumann universe and a specially-trimmed Boolean valued universe that are usually taken as these two models. Then we present a special ascending and descending machinery for interplay between the models. We consider the reals and complexes inside a Boolean valued model by using the celebrated Gordon’s Theorem which we read as follows: Every universally complete vector lattice is an interpretation of the reals in an appropriate Boolean-valued model. We proceed with demonstrating the Boolean valued approach to the two familiar problems: (1) When is a band preserving operator order bounded? (2) When is an order bounded operator a sum or difference of two lattice homomorphisms? In conclusion we briefly overview the details of some typical spaces and operators together with their Boolean valued representations.

1. BOOLEAN REQUISITES

We start with recalling some auxiliary facts about the construction and treatment of the von Neumann universe and a specially-trimmed Boolean valued universe.

Key words and phrases. Boolean valued analysis, vector lattice, positive operator.

1.1. The *von Neumann universe* \mathbb{V} results by transfinite recursion over ordinals. As the initial object of this construction we take the empty set. The elementary step of introducing new sets consists in uniting the powersets of the sets already available. Transfinitely repeating these steps, we exhaust the class of all sets. More precisely, we put $\mathbb{V} := \bigcup_{\alpha \in \text{On}} \mathbb{V}_\alpha$, where On is the class of all ordinals and

$$\begin{aligned} \mathbb{V}_0 &:= \emptyset, \\ \mathbb{V}_{\alpha+1} &:= \mathcal{P}(\mathbb{V}_\alpha), \\ \mathbb{V}_\beta &:= \bigcup_{\alpha < \beta} \mathbb{V}_\alpha \quad (\beta \text{ is a limit ordinal}). \end{aligned}$$

The class \mathbb{V} is the standard model of Zermelo–Fraenkel set theory.

1.2. Let \mathbb{B} be a complete Boolean algebra. Given an ordinal α , put

$$\begin{aligned} \mathbb{V}_\alpha^{(\mathbb{B})} &:= \{x : x \text{ is a function} \\ &\quad \wedge (\exists \beta)(\beta < \alpha \wedge \text{dom}(x) \subset \mathbb{V}_\beta^{(\mathbb{B})} \wedge \text{im}(x) \subset \mathbb{B})\}. \end{aligned}$$

After this recursive definition the *Boolean valued universe* $\mathbb{V}^{(\mathbb{B})}$ or, in other words, the *class of \mathbb{B} -sets* is introduced by

$$\mathbb{V}^{(\mathbb{B})} := \bigcup_{\alpha \in \text{On}} \mathbb{V}_\alpha^{(\mathbb{B})},$$

with On standing for the class of all ordinals.

In case of the two-element Boolean algebra $2 := \{0, 1\}$ this procedure yields a version of the classical von Neumann universe \mathbb{V} .

Let φ be an arbitrary formula of ZFC, Zermelo–Fraenkel set theory with choice. The *Boolean truth value* $\llbracket \varphi \rrbracket \in \mathbb{B}$ is introduced by induction on the length of a formula φ by using the natural interpretation of the propositional connectives and quantifiers in \mathbb{B} and the way in which φ results from atomic formulas. The Boolean truth values of the *atomic formulas* $x \in y$ and $x = y$, with $x, y \in \mathbb{V}^{(\mathbb{B})}$, are defined by means of the following recursion schema:

$$\begin{aligned} \llbracket x \in y \rrbracket &= \bigvee_{t \in \text{dom}(y)} y(t) \wedge \llbracket t = x \rrbracket, \\ \llbracket x = y \rrbracket &= \bigvee_{t \in \text{dom}(x)} x(t) \Rightarrow \llbracket t \in y \rrbracket \wedge \bigvee_{t \in \text{dom}(y)} y(t) \Rightarrow \llbracket t \in x \rrbracket. \end{aligned}$$

The sign \Rightarrow symbolizes the implication in \mathbb{B} ; i.e., $a \Rightarrow b := a^* \vee b$ where a^* is as usual the *complement* of a .

The universe $\mathbb{V}^{(\mathbb{B})}$ with the Boolean truth value of a formula is a model of set theory in the sense that every theorem of ZFC is true inside $\mathbb{V}^{(\mathbb{B})}$.

1.3. Transfer Principle. For every theorem φ of ZFC, we have $\llbracket \varphi \rrbracket = 1$; i.e., φ is true inside $\mathbb{V}^{(\mathbb{B})}$.

Enter into the next agreement: If x is an element of $\mathbb{V}^{(\mathbb{B})}$ and $\varphi(\cdot)$ is a formula of ZFC, then the phrase “ x satisfies φ inside $\mathbb{V}^{(\mathbb{B})}$ ” or, briefly, “ $\varphi(x)$ is true inside $\mathbb{V}^{(\mathbb{B})}$ ” means that $\llbracket \varphi(x) \rrbracket = 1$. This is sometimes written as $\mathbb{V}^{(\mathbb{B})} \models \varphi(x)$.

Given $x \in \mathbb{V}^{(\mathbb{B})}$ and $b \in \mathbb{B}$, define the function $bx : z \mapsto bx(z)$ ($z \in \text{dom}(x)$). Here we presume that $b\emptyset := \emptyset$ for all $b \in \mathbb{B}$.

There is a natural equivalence relation $x \sim y \leftrightarrow \llbracket x = y \rrbracket = 1$ in the class $\mathbb{V}^{(\mathbb{B})}$. Choosing a representative of the smallest rank in each equivalence class or,

more exactly, using the so-called “Frege–Russell–Scott trick,” we obtain a *separated Boolean valued universe* $\overline{\mathbb{V}}^{(\mathbb{B})}$ in which $x = y \leftrightarrow \llbracket x = y \rrbracket = 1$.

It is easily to see that the Boolean truth value of a formula remains unaltered if we replace in it each element of $\mathbb{V}^{(\mathbb{B})}$ by one of its equivalents. In this connection from now on we take $\mathbb{V}^{(\mathbb{B})} := \overline{\mathbb{V}}^{(\mathbb{B})}$ without further specification.

Observe that in $\overline{\mathbb{V}}^{(\mathbb{B})}$ the element bx is defined correctly for $x \in \overline{\mathbb{V}}^{(\mathbb{B})}$ and $b \in \mathbb{B}$ since $\llbracket x_1 = x_2 \rrbracket = 1 \rightarrow \llbracket bx_1 = bx_2 \rrbracket = b \Rightarrow \llbracket x_1 = x_2 \rrbracket = 1$. For a similar reason, we often write $\emptyset := \emptyset$, and in particular $\emptyset\emptyset = \emptyset = \emptyset x$ for $x \in \mathbb{V}^{(\mathbb{B})}$.

1.4. Mixing Principle. *Let $(b_\xi)_{\xi \in \Xi}$ be a partition of unity in \mathbb{B} , i.e. $\sup_{\xi \in \Xi} b_\xi = \sup \mathbb{B} = 1$ and $\xi \neq \eta \rightarrow b_\xi \wedge b_\eta = \emptyset$. To each family $(x_\xi)_{\xi \in \Xi}$ in $\mathbb{V}^{(\mathbb{B})}$ there exists a unique element x in the separated universe such that $\llbracket x = x_\xi \rrbracket \geq b_\xi$ ($\xi \in \Xi$).*

This element is called the *mixing* of $(x_\xi)_{\xi \in \Xi}$ by $(b_\xi)_{\xi \in \Xi}$ and is denoted by $\sum_{\xi \in \Xi} b_\xi x_\xi$. Thus, the mixing principle asserts that every Boolean valued universe is rich in mixings.

1.5. Maximum Principle. *The least upper bound is attained on the right-hand side of the formula for the Boolean truth-value of the existential quantifier. More precisely, if φ is a formula of ZFC then there is a \mathbb{B} -valued set x_0 satisfying $\llbracket (\exists x)\varphi(x) \rrbracket = \llbracket \varphi(x_0) \rrbracket$.*

2. THE ESCHER RULES

Boolean valued analysis consists primarily in comparison of the instances of a mathematical object or idea in two Boolean valued models. This is impossible to achieve without some dialog between the universes \mathbb{V} and $\mathbb{V}^{(\mathbb{B})}$. In other words, we need a smooth mathematical toolkit for revealing interplay between the interpretations of one and the same fact in the two models \mathbb{V} and $\mathbb{V}^{(\mathbb{B})}$. The relevant *ascending-and-descending technique* rests on the functors of canonical embedding, descent, and ascent.

2.1. We start with the canonical embedding of the von Neumann universe \mathbb{V} .

Given $x \in \mathbb{V}$, we denote by x^\wedge the *standard name* of x in $\mathbb{V}^{(\mathbb{B})}$; i.e., the element defined by the following recursion schema: $\emptyset^\wedge := \emptyset$, $\text{dom}(x^\wedge) := \{y^\wedge : y \in x\}$, $\text{im}(x^\wedge) := \{1\}$. Observe some properties of the mapping $x \mapsto x^\wedge$ we need in the sequel.

(1) For an arbitrary $x \in \mathbb{V}$ and a formula φ of ZFC we have

$$\begin{aligned} \llbracket (\exists y \in x^\wedge) \varphi(y) \rrbracket &= \bigvee_{z \in x} \llbracket \varphi(z^\wedge) \rrbracket, \\ \llbracket (\forall y \in x^\wedge) \varphi(y) \rrbracket &= \bigwedge_{z \in x} \llbracket \varphi(z^\wedge) \rrbracket. \end{aligned}$$

(2) If x and y are elements of \mathbb{V} then, by transfinite induction, we establish $x \in y \leftrightarrow \mathbb{V}^{(\mathbb{B})} \models x^\wedge \in y^\wedge$, $x = y \leftrightarrow \mathbb{V}^{(\mathbb{B})} \models x^\wedge = y^\wedge$. In other words, the standard name can be considered as an embedding of \mathbb{V} into $\mathbb{V}^{(\mathbb{B})}$. Moreover, it is beyond a doubt that the standard name sends \mathbb{V} onto $\mathbb{V}^{(2)}$, which fact is demonstrated by the next proposition:

(3) The following holds: $(\forall u \in \mathbb{V}^{(2)}) (\exists! x \in \mathbb{V}) \mathbb{V}^{(\mathbb{B})} \models u = x^\wedge$.

A formula is called *bounded* or *restricted* if each bound variable in it is restricted by a bounded quantifier; i.e., a quantifier ranging over a particular set. The latter

means that each bound variable x is restricted by a quantifier of the form $(\forall x \in y)$ or $(\exists x \in y)$ for some y .

2.2. Restricted Transfer Principle. *For each bounded formula φ of ZFC and every collection $x_1, \dots, x_n \in \mathbb{V}$ the following holds: $\varphi(x_1, \dots, x_n) \leftrightarrow \mathbb{V}^{(\mathbb{B})} \models \varphi(x_1^\wedge, \dots, x_n^\wedge)$. Henceforth, working in the separated universe $\overline{\mathbb{V}}^{(\mathbb{B})}$, we agree to preserve the symbol x^\wedge for the distinguished element of the class corresponding to x .*

Observe for example that the restricted transfer principle yields:

$$\begin{aligned} & \text{“}\Phi \text{ is a correspondence from } x \text{ to } y\text{”} \\ & \leftrightarrow \mathbb{V}^{(\mathbb{B})} \models \text{“}\Phi^\wedge \text{ is a correspondence from } x^\wedge \text{ to } y^\wedge\text{”}; \\ & \text{“}f : x \rightarrow y\text{”} \leftrightarrow \mathbb{V}^{(\mathbb{B})} \models \text{“}f^\wedge : x^\wedge \rightarrow y^\wedge\text{”} \end{aligned}$$

(moreover, $f(a)^\wedge = f^\wedge(a^\wedge)$ for all $a \in x$). Thus, the standard name can be considered as a covariant functor of the category of sets (or correspondences) inside \mathbb{V} to an appropriate subcategory of $\mathbb{V}^{(2)}$ in the separated universe $\mathbb{V}^{(\mathbb{B})}$.

2.3. A set X is *finite* if X coincides with the image of a function on a finite ordinal. In symbols, this is expressed as $\text{fin}(X)$; hence,

$$\text{fin}(X) := (\exists n)(\exists f)(n \in \omega \wedge f \text{ is a function} \wedge \text{dom}(f) = n \wedge \text{im}(f) = X)$$

(as usual $\omega := \{0, 1, 2, \dots\}$). Obviously, the above formula is not bounded. Nevertheless there is a simple transformation rule for the class of finite sets under the canonical embedding. Denote by $\mathcal{P}_{\text{fin}}(X)$ the class of all finite subsets of X ; i.e., $\mathcal{P}_{\text{fin}}(X) := \{Y \in \mathcal{P}(X) : \text{fin}(Y)\}$. For an arbitrary set X the following holds: $\mathbb{V}^{(\mathbb{B})} \models \mathcal{P}_{\text{fin}}(X)^\wedge = \mathcal{P}_{\text{fin}}(X^\wedge)$.

2.4. Given an arbitrary element x of the (separated) Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$, we define the *descent* $x \downarrow$ of x as $x \downarrow := \{y \in \mathbb{V}^{(\mathbb{B})} : \llbracket y \in x \rrbracket = 1\}$. We list the simplest properties of descending:

(1) The class $x \downarrow$ is a set, i.e., $x \downarrow \in \mathbb{V}$ for all $x \in \mathbb{V}^{(\mathbb{B})}$. If $\llbracket x \neq \emptyset \rrbracket = 1$ then $x \downarrow$ is a nonempty set.

(2) Let $z \in \mathbb{V}^{(\mathbb{B})}$ and $\llbracket z \neq \emptyset \rrbracket = 1$. Then for every formula φ of ZFC we have

$$\begin{aligned} \llbracket (\forall x \in z) \varphi(x) \rrbracket &= \bigwedge_{x \in z \downarrow} \llbracket \varphi(x) \rrbracket, \\ \llbracket (\exists x \in z) \varphi(x) \rrbracket &= \bigvee_{x \in z \downarrow} \llbracket \varphi(x) \rrbracket. \end{aligned}$$

Moreover, there exists $x_0 \in z \downarrow$ such that $\llbracket \varphi(x_0) \rrbracket = \llbracket (\exists x \in z) \varphi(x) \rrbracket$.

(3) Let Φ be a correspondence from X to Y in $\mathbb{V}^{(\mathbb{B})}$. Thus, Φ , X , and Y are elements of $\mathbb{V}^{(\mathbb{B})}$ and, moreover, $\llbracket \Phi \subset X \times Y \rrbracket = 1$. There is a unique correspondence $\Phi \downarrow$ from $X \downarrow$ to $Y \downarrow$ such that $\Phi \downarrow(A \downarrow) = \Phi(A) \downarrow$ for every nonempty subset A of X inside $\mathbb{V}^{(\mathbb{B})}$. The correspondence $\Phi \downarrow$ from $X \downarrow$ to $Y \downarrow$ of the above proposition is called the *descent* of the correspondence Φ from X to Y inside $\mathbb{V}^{(\mathbb{B})}$.

(4) The descent of the composite of correspondences inside $\mathbb{V}^{(\mathbb{B})}$ is the composite of their descents: $(\Psi \circ \Phi) \downarrow = \Psi \downarrow \circ \Phi \downarrow$.

(5) If Φ is a correspondence inside $\mathbb{V}^{(\mathbb{B})}$ then $(\Phi^{-1}) \downarrow = (\Phi \downarrow)^{-1}$.

(6) Let Id_X be the identity mapping inside $\mathbb{V}^{(\mathbb{B})}$ of a set $X \in \mathbb{V}^{(\mathbb{B})}$. Then $(\text{Id}_X) \downarrow = \text{Id}_{X \downarrow}$.

(7) Suppose that $X, Y, f \in \mathbb{V}^{(\mathbb{B})}$ are such that $\llbracket f : X \rightarrow Y \rrbracket = 1$, i.e., f is a mapping from X to Y inside $\mathbb{V}^{(\mathbb{B})}$. Then $f \downarrow$ is a unique mapping from $X \downarrow$ to $Y \downarrow$ satisfying $\llbracket f \downarrow(x) = f(x) \rrbracket = 1$ for all $x \in X \downarrow$.

By virtue of (1)–(7), we can consider the descent operation as a functor from the category of \mathbb{B} -valued sets and mappings (correspondences) to the category of the usual sets and mappings (correspondences) (i.e., in the sense of \mathbb{V}).

(8) Given $x_1, \dots, x_n \in \mathbb{V}^{(\mathbb{B})}$, denote by $(x_1, \dots, x_n)^{\mathbb{B}}$ the corresponding ordered n -tuple inside $\mathbb{V}^{(\mathbb{B})}$. Assume that P is an n -ary relation on X inside $\mathbb{V}^{(\mathbb{B})}$; i.e., $X, P \in \mathbb{V}^{(\mathbb{B})}$ and $\llbracket P \subset X^{n^{\wedge}} \rrbracket = 1$, where $n \in \omega$. Then there exists an n -ary relation P' on $X \downarrow$ such that $(x_1, \dots, x_n) \in P' \leftrightarrow \llbracket (x_1, \dots, x_n)^{\mathbb{B}} \in P \rrbracket = 1$. Slightly abusing notation, we denote the relation P' by the same symbol $P \downarrow$ and call it the *descent* of P .

2.5. Let $x \in \mathbb{V}$ and $x \subset \mathbb{V}^{(\mathbb{B})}$; i.e., let x be some set composed of \mathbb{B} -valued sets or, in other words, $x \in \mathcal{P}(\mathbb{V}^{(\mathbb{B})})$. Put $\emptyset \uparrow := \emptyset$ and $\text{dom}(x \uparrow) := x$, $\text{im}(x \uparrow) := \{1\}$ if $x \neq \emptyset$. The element $x \uparrow$ (of the separated universe $\mathbb{V}^{(\mathbb{B})}$, i.e., the distinguished representative of the class $\{y \in \mathbb{V}^{(\mathbb{B})} : \llbracket y = x \uparrow \rrbracket = 1\}$) is called the *ascent* of x .

(1) For all $x \in \mathcal{P}(\mathbb{V}^{(\mathbb{B})})$ and every formula φ we have the following:

$$\begin{aligned} \llbracket (\forall z \in x \uparrow) \varphi(z) \rrbracket &= \bigwedge_{y \in x} \llbracket \varphi(y) \rrbracket, \\ \llbracket (\exists z \in x \uparrow) \varphi(z) \rrbracket &= \bigvee_{y \in x} \llbracket \varphi(y) \rrbracket. \end{aligned}$$

Introducing the ascent of a correspondence $\Phi \subset X \times Y$, we have to bear in mind a possible distinction between the domain of departure X and the domain $\text{dom}(\Phi) := \{x \in X : \Phi(x) \neq \emptyset\}$. This circumstance is immaterial for the sequel; therefore, speaking of ascents, we always imply total correspondences; i.e., $\text{dom}(\Phi) = X$.

(2) Let $X, Y, \Phi \in \mathbb{V}^{(\mathbb{B})}$, and let Φ be a correspondence from X to Y . There exists a unique correspondence $\Phi \uparrow$ from $X \uparrow$ to $Y \uparrow$ inside $\mathbb{V}^{(\mathbb{B})}$ such that $\Phi \uparrow(A \uparrow) = \Phi(A) \uparrow$ is valid for every subset A of $\text{dom}(\Phi)$ if and only if Φ is *extensional*; i.e., satisfies the condition $y_1 \in \Phi(x_1) \rightarrow \llbracket x_1 = x_2 \rrbracket \leq \bigvee_{y_2 \in \Phi(x_2)} \llbracket y_1 = y_2 \rrbracket$ for $x_1, x_2 \in \text{dom}(\Phi)$. In this event, $\Phi \uparrow = \Phi' \uparrow$, where $\Phi' := \{(x, y)^{\mathbb{B}} : (x, y) \in \Phi\}$. The element $\Phi \uparrow$ is called the *ascent* of Φ .

(3) The composite of extensional correspondences is extensional. Moreover, the ascent of a composite is equal to the composite of the ascents inside $\mathbb{V}^{(\mathbb{B})}$: On assuming that $\text{dom}(\Psi) \supset \text{im}(\Phi)$ we have $\mathbb{V}^{(\mathbb{B})} \models (\Psi \circ \Phi) \uparrow = \Psi \uparrow \circ \Phi \uparrow$.

Note that if Φ and Φ^{-1} are extensional then $(\Phi \uparrow)^{-1} = (\Phi^{-1} \uparrow)$. However, in general, the extensionality of Φ in no way guarantees the extensionality of Φ^{-1} .

(4) It is worth mentioning that if an extensional correspondence f is a function from X to Y then the ascent $f \uparrow$ of f is a function from $X \uparrow$ to $Y \uparrow$. Moreover, the extensionality property can be stated as follows: $\llbracket x_1 = x_2 \rrbracket \leq \llbracket f(x_1) = f(x_2) \rrbracket$ for all $x_1, x_2 \in X$.

2.6. Given a set $X \subset \mathbb{V}^{(\mathbb{B})}$, we denote by the symbol $\text{mix}(X)$ the set of all mixings of the form $\text{mix}(b_{\xi} x_{\xi})$, where $(x_{\xi}) \subset X$ and (b_{ξ}) is an arbitrary partition of unity. The following propositions are referred to as the *arrow cancellation rules* or *ascending-and-descending rules*. There are many good reasons to call them simply the *Escher rules* [19].

(1) Let X and X' be subsets of $\mathbb{V}^{(\mathbb{B})}$ and let $f : X \rightarrow X'$ be an extensional mapping. Suppose that $Y, Y', g \in \mathbb{V}^{(\mathbb{B})}$ are such that $\llbracket Y \neq \emptyset \rrbracket = \llbracket g : Y \rightarrow Y' \rrbracket = 1$. Then $X \uparrow \downarrow = \text{mix}(X), Y \downarrow \uparrow = Y, f \uparrow \downarrow = f$, and $g \downarrow \uparrow = g$.

(2) From 2.3 (8) we easily infer the useful relation: $\mathcal{P}_{\text{fin}}(X \uparrow) = \{\theta \uparrow : \theta \in \mathcal{P}_{\text{fin}}(X)\} \uparrow$.

Suppose that $X \in \mathbb{V}$, $X \neq \emptyset$; i.e., X is a nonempty set. Let the letter ι denote the standard name embedding $x \mapsto x^\wedge$ ($x \in X$). Then $\iota(X) \uparrow = X^\wedge$ and $X = \iota^{-1}(X^\wedge \downarrow)$. Using the above relations, we may extend the descent and ascent operations to the case in which Φ is a correspondence from X to $Y \downarrow$ and $\llbracket \Psi \rrbracket$ is a correspondence from X^\wedge to $Y \uparrow = 1$, where $Y \in \mathbb{V}^{(\mathbb{B})}$. Namely, we put $\Phi \uparrow := (\Phi \circ \iota) \uparrow$ and $\Psi \downarrow := \Psi \downarrow \circ \iota$. In this case, $\Phi \uparrow$ is called the *modified ascent* of Φ and $\Psi \downarrow$ is called the *modified descent* of Ψ . (If the context excludes ambiguity then we briefly speak of ascents and descents using simple arrows.) It is easy to see that $\Psi \uparrow$ is a unique correspondence inside $\mathbb{V}^{(\mathbb{B})}$ satisfying the relation $\llbracket \Phi \uparrow(x^\wedge) = \Phi(x) \uparrow \rrbracket = 1$ ($x \in X$). Similarly, $\Psi \downarrow$ is a unique correspondence from X to $Y \downarrow$ satisfying the equality $\Psi \downarrow(x) = \Psi(x^\wedge) \downarrow$ ($x \in X$). If $\Phi := f$ and $\Psi := g$ are functions then these relations take the form $\llbracket f \uparrow(x^\wedge) = f(x) \rrbracket = 1$ and $g \downarrow(x) = g(x^\wedge)$ for all $x \in X$.

2.7. Various function spaces reside in functional analysis, and so the problem is natural of replacing an abstract Boolean valued system by some function-space analog, a model whose elements are functions and in which the basic logical operations are calculated “pointwise.” An example of such a model is given by the class \mathbb{V}^Q of all functions defined on a fixed nonempty set Q and acting into \mathbb{V} . The truth values on \mathbb{V}^Q are various subsets of Q : The truth value $\llbracket \varphi(u_1, \dots, u_n) \rrbracket$ of $\varphi(t_1, \dots, t_n)$ at functions $u_1, \dots, u_n \in \mathbb{V}^Q$ is calculated as follows:

$$\llbracket \varphi(u_1, \dots, u_n) \rrbracket = \{q \in Q : \varphi(u_1(q), \dots, u_n(q))\}.$$

A. G. Gutman and G. A. Losenkov solved the above problem by the concept of continuous polyverse which is a continuous bundle of models of set theory. It is shown that the class of continuous sections of a continuous polyverse is a Boolean valued system satisfying all basic principles of Boolean valued analysis and, conversely, each Boolean valued algebraic system can be represented as the class of sections of a suitable continuous polyverse. More details are collected in [37, Chapter 6].

3. BOOLEAN VALUED ALGEBRAIC SYSTEMS

Every Boolean valued universe has the collection of mathematical objects in full supply: available in plenty are all sets with extra structure: groups, rings, algebras, normed spaces, etc. Applying the descent functor to such *internal* algebraic systems of a Boolean valued model, we distinguish some bizarre entities or recognize old acquaintances, which leads to revealing the new facts of their life and structure.

This technique of research, known as *direct Boolean valued interpretation*, allows us to produce new theorems or, to be more exact, to extend the semantical content of the available theorems by means of slavish translation. The information we so acquire might fail to be vital, valuable, or intriguing, in which case the direct Boolean valued interpretation turns out into a leisurely game.

It thus stands to reason to raise the following questions: What structures significant for mathematical practice are obtainable by the Boolean valued interpretation

of the most typical algebraic systems? What transfer principles hold true in this process? Clearly, the answers should imply specific objects whose particular features enable us to deal with their Boolean valued representation which, if understood duly, is impossible to implement for arbitrary algebraic systems.

3.1. An *abstract Boolean set* or *set with \mathbb{B} -structure* is a pair (X, d) , where $X \in \mathbb{V}$, $X \neq \emptyset$, and d is a mapping from $X \times X$ to \mathbb{B} such that $d(x, y) = 0 \leftrightarrow x = y$; $d(x, y) = d(y, x)$; $d(x, y) \leq d(x, z) \vee d(z, y)$ all $x, y, z \in X$.

To obtain an easy example of an abstract \mathbb{B} -set, given $\emptyset \neq X \subset \mathbb{V}^{(\mathbb{B})}$ put

$$d(x, y) := \llbracket x \neq y \rrbracket = \neg \llbracket x = y \rrbracket$$

for $x, y \in X$.

Another easy example is a nonempty X with the *discrete \mathbb{B} -metric d* ; i.e., $d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$.

3.2. Let (X, d) be some abstract \mathbb{B} -set. There exist an element $\mathcal{X} \in \mathbb{V}^{(\mathbb{B})}$ and an injection $\iota : X \rightarrow X' := \mathcal{X} \downarrow$ such that $d(x, y) = \llbracket \iota x \neq \iota y \rrbracket$ for all $x, y \in X$ and every element $x' \in X'$ admits the representation $x' = \text{mix}_{\xi \in \Xi} (b_\xi \iota x_\xi)$, where $(x_\xi)_{\xi \in \Xi} \subset X$ and $(b_\xi)_{\xi \in \Xi}$ is a partition of unity in \mathbb{B} .

We see that an abstract \mathbb{B} -set X embeds in the Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$ so that the Boolean distance between the members of X becomes the Boolean truth value of the negation of their equality. The corresponding element $\mathcal{X} \in \mathbb{V}^{(\mathbb{B})}$ is, by definition, the *Boolean valued representation* of X .

If X is a discrete abstract \mathbb{B} -set then $\mathcal{X} = X^\wedge$ and $\iota x = x^\wedge$ for all $x \in X$. If $X \subset \mathbb{V}^{(\mathbb{B})}$ then $\iota \uparrow$ is an injection from $X \uparrow$ to \mathcal{X} (inside $\mathbb{V}^{(\mathbb{B})}$).

3.3. A mapping f from a \mathbb{B} -set (X, d) to a \mathbb{B} -set (X', d') is said to be *contractive* if $d(x, y) \geq d'(f(x), f(y))$ for all $x, y \in X$.

Let X and Y be some \mathbb{B} -sets, \mathcal{X} and \mathcal{Y} be their Boolean-value representations, and ι and \varkappa be the corresponding injections $X \rightarrow \mathcal{X} \downarrow$ and $Y \rightarrow \mathcal{Y} \downarrow$. If $f : X \rightarrow Y$ is a contractive mapping then there is a unique element $g \in \mathbb{V}^{(\mathbb{B})}$ such that $\llbracket g : \mathcal{X} \rightarrow \mathcal{Y} \rrbracket = 1$ and $f = \varkappa^{-1} \circ g \downarrow \circ \iota$. We also accept the notations $\mathcal{X} := \mathcal{F}^\sim(X) := X^\sim$ and $g := \mathcal{F}^\sim(f) := f^\sim$.

3.4. The following are valid:

- (1) $\mathbb{V}^{(\mathbb{B})} \models f(A)^\sim = f^\sim(A^\sim)$ for $A \subset X$;
- (2) If $g : Y \rightarrow Z$ is a contraction then $g \circ f$ is a contraction and $\mathbb{V}^{(\mathbb{B})} \models (g \circ f)^\sim = g^\sim \circ f^\sim$;
- (3) $\mathbb{V}^{(\mathbb{B})} \models "f^\sim \text{ is injective}"$ if and only if f is a \mathbb{B} -isometry;
- (4) $\mathbb{V}^{(\mathbb{B})} \models "f^\sim \text{ is surjective}"$ if and only if $\bigvee \{d(f(x), y) : x \in X\} = 1$ for every $y \in Y$.

3.5. In case a \mathbb{B} -set X has some a priori structure we may try to furnish the Boolean valued representation of X with an analogous structure, so as to apply the technique of ascending and descending to the study of the original structure of X . Consequently, the above questions may be treated as instances of the unique problem of searching a well-qualified Boolean valued representation of a \mathbb{B} -set with some additional structure. We call these objects *algebraic \mathbb{B} -systems*.

Recall that a *signature* is a 3-tuple $\sigma := (F, P, \mathfrak{a})$, where F and P are some (possibly, empty) sets and \mathfrak{a} is a mapping from $F \cup P$ to ω . If the sets F and P are finite then σ is a *finite signature*. In applications we usually deal with algebraic systems of finite signature.

An n -ary operation and an n -ary predicate on a \mathbb{B} -set A are contractive mappings $f : A^n \rightarrow A$ and $p : A^n \rightarrow \mathbb{B}$ respectively. By definition, f and p are *contractive mappings* provided that

$$d(f(a_0, \dots, a_{n-1}), f(a'_0, \dots, a'_{n-1})) \leq \bigvee_{k=0}^{n-1} d(a_k, a'_k),$$

$$d_s(p(a_0, \dots, a_{n-1}), p(a'_0, \dots, a'_{n-1})) \leq \bigvee_{k=0}^{n-1} d(a_k, a'_k)$$

for all $a_0, a'_0, \dots, a_{n-1}, a'_{n-1} \in A$, where d is the \mathbb{B} -metric of A , and d_s is the *symmetric difference* on \mathbb{B} ; i.e., $d_s(b_1, b_2) := b_1 \triangle b_2$ (cf. 1.1.4).

Clearly, the above definitions depend on \mathbb{B} and it would be cleaner to speak of \mathbb{B} -operations, \mathbb{B} -predicates, etc. We adhere to a simpler practice whenever it entails no confusion.

3.6. An *algebraic \mathbb{B} -system* \mathfrak{A} of signature σ is a pair (A, ν) , where A is a nonempty \mathbb{B} -set, the *underlying set*, or *carrier*, or *universe* of \mathfrak{A} , and ν is a mapping such that (a) $\text{dom}(\nu) = F \cup P$; (b) $\nu(f)$ is an $\mathfrak{a}(f)$ -ary operation on A for all $f \in F$; and (c) $\nu(p)$ is an $\mathfrak{a}(p)$ -ary predicate on A for every $p \in P$.

It is in common parlance to call ν the *interpretation* of \mathfrak{A} , in which case the notation f^ν and p^ν are common substitutes for $\nu(f)$ and $\nu(p)$.

The signature of an algebraic \mathbb{B} -system $\mathfrak{A} := (A, \nu)$ is often denoted by $\sigma(\mathfrak{A})$; while the carrier A of \mathfrak{A} , by $|\mathfrak{A}|$. Since $A^0 = \{\emptyset\}$, the nullary operations and predicates on A are mappings from $\{\emptyset\}$ to A and \mathbb{B} respectively. We agree to identify a mapping $g : \{\emptyset\} \rightarrow A \cup \mathbb{B}$ with the element $g(\emptyset)$. Each nullary operation on A thus transforms into a unique member of A . Analogously, the set of all nullary predicates on A turns into the Boolean algebra \mathbb{B} . If $F := \{f_1, \dots, f_n\}$ and $P := \{p_1, \dots, p_m\}$ then an algebraic \mathbb{B} -system of signature σ is often written down as $(A, \nu(f_1), \dots, \nu(f_n), \nu(p_1), \dots, \nu(p_m))$ or even $(A, f_1, \dots, f_n, p_1, \dots, p_m)$. In this event, the expression $\sigma = (f_1, \dots, f_n, p_1, \dots, p_m)$ is substituted for $\sigma = (F, P, \mathfrak{a})$.

3.7. We now address the \mathbb{B} -valued interpretation of a first-order language. Consider an algebraic \mathbb{B} -system $\mathfrak{A} := (A, \nu)$ of signature $\sigma := \sigma(\mathfrak{A}) := (F, P, \mathfrak{a})$. Let $\varphi(x_0, \dots, x_{n-1})$ be a formula of signature σ with n free variables. Assume given $a_0, \dots, a_{n-1} \in A$. We may readily define the truth value $|\varphi|^\mathfrak{A}(a_0, \dots, a_{n-1}) \in \mathbb{B}$ of a formula φ in the system \mathfrak{A} for the given values a_0, \dots, a_{n-1} of the variables x_0, \dots, x_{n-1} . The definition proceeds as usual by induction on the complexity of φ : Considering propositional connectives and quantifiers, we put

$$|\varphi \wedge \psi|^\mathfrak{A}(a_0, \dots, a_{n-1}) := |\varphi|^\mathfrak{A}(a_0, \dots, a_{n-1}) \wedge |\psi|^\mathfrak{A}(a_0, \dots, a_{n-1});$$

$$|\varphi \vee \psi|^\mathfrak{A}(a_0, \dots, a_{n-1}) := |\varphi|^\mathfrak{A}(a_0, \dots, a_{n-1}) \vee |\psi|^\mathfrak{A}(a_0, \dots, a_{n-1});$$

$$|\neg \varphi|^\mathfrak{A}(a_0, \dots, a_{n-1}) := |\varphi|^\mathfrak{A}(a_0, \dots, a_{n-1})^*;$$

$$|(\forall x_0) \varphi|^\mathfrak{A}(a_1, \dots, a_{n-1}) := \bigwedge_{a_0 \in A} |\varphi|^\mathfrak{A}(a_0, \dots, a_{n-1});$$

$$|(\exists x_0) \varphi|^\mathfrak{A}(a_1, \dots, a_{n-1}) := \bigvee_{a_0 \in A} |\varphi|^\mathfrak{A}(a_0, \dots, a_{n-1}).$$

Now, the case of atomic formulas is in order. Assume that $p \in P$ symbolizes an m -ary predicate, $q \in P$ is a nullary predicate, and t_0, \dots, t_{m-1} be terms of signature

σ assuming values b_0, \dots, b_{m-1} at the given values a_0, \dots, a_{n-1} of the variables x_0, \dots, x_{n-1} . By definition, we let

$$\begin{aligned} |\varphi|^{\mathfrak{A}}(a_0, \dots, a_{n-1}) &:= \nu(q), \text{ if } \varphi = q^\nu; \\ |\varphi|^{\mathfrak{A}}(a_0, \dots, a_{n-1}) &:= d(b_0, b_1)^*, \text{ if } \varphi = (t_0 = t_1); \\ |\varphi|^{\mathfrak{A}}(a_0, \dots, a_{n-1}) &:= p^\nu(b_0, \dots, b_{m-1}), \text{ if } \varphi = p^\nu(t_0, \dots, t_{m-1}), \end{aligned}$$

where d is a \mathbb{B} -metric on A .

Say that $\varphi(x_0, \dots, x_{n-1})$ is *valid* in \mathfrak{A} at the given values $a_0, \dots, a_{n-1} \in A$ of x_0, \dots, x_{n-1} and write $\mathfrak{A} \models \varphi(a_0, \dots, a_{n-1})$ provided that $|\varphi|^{\mathfrak{A}}(a_0, \dots, a_{n-1}) = 1_{\mathbb{B}}$. Alternative expressions are as follows: $a_0, \dots, a_{n-1} \in A$ *satisfies* $\varphi(x_0, \dots, x_{n-1})$; or $\varphi(a_0, \dots, a_{n-1})$ holds true in \mathfrak{A} . In case $\mathbb{B} := \{0, 1\}$, we arrive at the conventional definition of the validity of a formula in an algebraic system.

Recall that a closed formula φ of signature σ is *tautology* if φ is valid on every algebraic 2-system of signature σ .

3.8. Before giving a general definition of the descent of an algebraic system, consider the descent of a very simple but important algebraic system, the two-element Boolean algebra. Choose two arbitrary elements, $0, 1 \in \mathbb{V}^{(\mathbb{B})}$, satisfying $\llbracket 0 \neq 1 \rrbracket = 1_{\mathbb{B}}$. We may for instance assume that $0 := 0_{\mathbb{B}}$ and $1 := 1_{\mathbb{B}}$.

The descent C of the two-element Boolean algebra $\{0, 1\}^{\mathbb{B}} \in \mathbb{V}^{(\mathbb{B})}$ is a complete Boolean algebra isomorphic to \mathbb{B} . The formulas

$$\llbracket \chi(b) = 1 \rrbracket = b, \quad \llbracket \chi(b) = 0 \rrbracket = b^* \quad (b \in \mathbb{B})$$

defines an isomorphism $\chi : \mathbb{B} \rightarrow C$.

3.9. Consider now an algebraic system \mathfrak{A} of signature σ^\wedge inside $\mathbb{V}^{(\mathbb{B})}$, and let $\llbracket \mathfrak{A} = (A, \nu)^{\mathbb{B}} \rrbracket = 1$ for some $A, \nu \in \mathbb{V}^{(\mathbb{B})}$. The *descent* of \mathfrak{A} is the pair $\mathfrak{A} \downarrow := (A \downarrow, \mu)$, where μ is the function determined from the formulas:

$$\begin{aligned} \mu : f &\mapsto (\nu \downarrow(f)) \downarrow \quad (f \in F), \\ \mu : p &\mapsto \chi^{-1} \circ (\nu \downarrow(p)) \downarrow \quad (p \in P). \end{aligned}$$

Here χ is the above-defined isomorphism of \mathbb{B} .

In more detail, the modified descent $\nu \downarrow$ is the mapping with domain $\text{dom}(\nu \downarrow) = F \cup P$. Given $p \in P$, observe $\llbracket \mathfrak{a}(p)^\wedge = \mathfrak{a}^\wedge(p^\wedge) \rrbracket = 1$, $\llbracket \nu \downarrow(p) = \nu(p^\wedge) \rrbracket = 1$ and so

$$\mathbb{V}^{(\mathbb{B})} \models \nu \downarrow(p) : A^{\mathfrak{a}(f)^\wedge} \rightarrow \{0, 1\}^{\mathbb{B}}.$$

It is now obvious that $(\nu \downarrow(p)) \downarrow : (A \downarrow)^{\mathfrak{a}(f)} \rightarrow C := \{0, 1\}^{\mathbb{B}} \downarrow$ and we may put $\mu(p) := \chi^{-1} \circ (\nu \downarrow(p)) \downarrow$.

3.10. Let $\varphi(x_0, \dots, x_{n-1})$ be a fixed formula of signature σ in n free variables. Write down the formula $\Phi(x_0, \dots, x_{n-1}, \mathfrak{A})$ in the language of set theory which formalizes the proposition $\mathfrak{A} \models \varphi(x_0, \dots, x_{n-1})$. Recall that the formula $\mathfrak{A} \models \varphi(x_0, \dots, x_{n-1})$ determines an n -ary predicate on A or, which is the same, a mapping from A^n to $\{0, 1\}$. By the maximum and transfer principles, there is a unique element $|\varphi|^{\mathfrak{A}} \in \mathbb{V}^{(\mathbb{B})}$ such that

$$\llbracket |\varphi|^{\mathfrak{A}} : A^{n^\wedge} \rightarrow \{0, 1\}^{\mathbb{B}} \rrbracket = 1,$$

$$\llbracket |\varphi|^{\mathfrak{A}}(a \uparrow) = 1 \rrbracket = \llbracket \Phi(a(0), \dots, a(n-1), \mathfrak{A}) \rrbracket = 1$$

for every $a : n \rightarrow A \downarrow$. Henceforth instead of $|\varphi|^{\mathfrak{A}} (a \uparrow)$ we will write $|\varphi|^{\mathfrak{A}} (a_0, \dots, a_{n-1})$, where $a_l := a(l)$. Therefore, the formula

$$\mathbb{V}^{(\mathbb{B})} \models \text{“}\varphi(a_0, \dots, a_{n-1}) \text{ is valid in } \mathfrak{A}\text{”}$$

holds true if and only if $\llbracket \Phi(a_0, \dots, a_{n-1}, \mathfrak{A}) \rrbracket = 1$.

Let \mathfrak{A} be an algebraic system of signature σ^\wedge inside $\mathbb{V}^{(\mathbb{B})}$. Then $\mathfrak{A} \downarrow$ is a universally complete algebraic \mathbb{B} -system of signature σ . In this event,

$$\chi \circ |\varphi|^{\mathfrak{A} \downarrow} = |\varphi|^{\mathfrak{A}} \downarrow.$$

for each formula φ of signature σ .

3.11. Let $\mathfrak{A} := (A, \nu)$ be an algebraic \mathbb{B} -system of signature σ . Then there are \mathcal{A} and $\mu \in \mathbb{V}^{(\mathbb{B})}$ such that the following are fulfilled:

- (1) $\mathbb{V}^{(\mathbb{B})} \models \text{“}(\mathcal{A}, \mu) \text{ is an algebraic system of signature } \sigma^\wedge\text{”}$;
- (2) If $\mathfrak{A}' := (A', \nu')$ is the descent of (\mathcal{A}, μ) then \mathfrak{A}' is a universally complete algebraic \mathbb{B} -system of signature σ ;
- (3) There is an isomorphism ι from \mathfrak{A} to \mathfrak{A}' such that $A' = \text{mix}(\iota(A))$;
- (4) For every formula φ of signature σ in n free variables, the equalities hold

$$\begin{aligned} |\varphi|^{\mathfrak{A}}(a_0, \dots, a_{n-1}) &= |\varphi|^{\mathfrak{A}'}(\iota(a_0), \dots, \iota(a_{n-1})) \\ &= \chi^{-1} \circ (|\varphi|^{\mathfrak{A}'} \downarrow)(\iota(a_0), \dots, \iota(a_{n-1})) \end{aligned}$$

for all $a_0, \dots, a_{n-1} \in A$ and χ the same as in 3.8.

3.12. The Boolean valued representation of an algebraic \mathbb{B} -system appears to be a conventional two-valued algebraic system of the same type. This means that an appropriate completion of each algebraic \mathbb{B} -system coincides with the descent of some two-valued algebraic system inside $\mathbb{V}^{(\mathbb{B})}$.

On the other hand, each two-valued algebraic system may be transformed into an algebraic \mathbb{B} -system on distinguishing a complete Boolean algebra of congruences of the original system. In this event, the task is in order of finding the formulas holding true in direct or reverse transition from a \mathbb{B} -system to a two-valued system. In other words, we have to seek here for some versions of the transfer or identity preservation principle of long standing in some branches of mathematics.

4. BOOLEAN VALUED NUMBERS

Boolean valued analysis stems from the fact that each internal field of reals of a Boolean valued model descends into a universally complete Kantorovich space. Thus, a remarkable opportunity opens up to expand and enrich the treasure-trove of mathematical knowledge by translating information about the reals to the language of other noble families of functional analysis. We will elaborate upon the matter in this section.

4.1. Recall a few definitions. A *vector lattice* is an ordered vector space whose order makes it a lattice. In other words, the *join* $\sup\{x_1, \dots, x_n\} := x_1 \vee \dots \vee x_n$ and *meet* $\inf\{x_1, \dots, x_n\} := x_1 \wedge \dots \wedge x_n$ correspond to each finite subset $\{x_1, \dots, x_n\}$ of a vector lattice. In particular, each element x has the *positive part* $x^+ := x \vee 0$, *negative part* $x^- := (-x)^+ := -x \wedge 0$, and *modulus* $|x| := x \vee (-x)$.

A vector lattice E is called *Archimedean* if for every pair of elements $x, y \in E$ from $(\forall n \in \mathbb{N}) \ nx \leq y$ it follows that $x \leq 0$. We assume all vector lattices Archimedean in what follows.

Two elements x and y of a vector lattice E are *disjoint* (in symbols $x \perp y$) if $|x| \wedge |y| = 0$. A *band* of E is defined as the *disjoint complement* $M^\perp := \{x \in E : (\forall y \in M) x \perp y\}$ of a nonempty set $M \subset E$.

The inclusion-ordered set $\mathfrak{B}(E)$ of all bands in E is a complete Boolean algebra with the Boolean operations:

$$L \wedge K = L \cap K, \quad L \vee K = (L \cup K)^{\perp\perp}, \quad L^* = L^\perp \quad (L, K \in \mathfrak{B}(E)).$$

The Boolean algebra $\mathfrak{B}(E)$ is often referred as to the *base* of E .

A *band projection* in E is a linear idempotent operator in $\pi : E \rightarrow E$ satisfying the inequalities $0 \leq \pi x \leq x$ for all $0 \leq x \in E$. The set $\mathfrak{P}(E)$ of all band projections ordered by $\pi \leq \rho \iff \pi \circ \rho = \pi$ is a Boolean algebra with the Boolean operations:

$$\pi \wedge \rho = \pi \circ \rho, \quad \pi \vee \rho = \pi + \rho - \pi \circ \rho, \quad \pi^* = I_E - \pi \quad (\pi, \rho \in \mathfrak{P}(E)).$$

Let $u \in E_+$ and $e \wedge (u - e) = 0$ for some $0 \leq e \in E$. Then e is a *fragment* or *component* of u . The set $\mathfrak{E}(u)$ of all fragments of u with the order induced by E is a Boolean algebra where the lattice operations are taken from E and the Boolean complement has the form $e^* := u - e$.

4.2. A Dedekind complete vector lattice is also called a *Kantorovich space* or *K-space*, for short. A *K-space* E is *universally complete* if every family of pairwise disjoint elements of E is order bounded.

(1) Theorem. *Let E be an arbitrary K-space. Then the correspondence $\pi \mapsto \pi(E)$ determines an isomorphism of the Boolean algebras $\mathfrak{P}(E)$ and $\mathfrak{B}(E)$. If there is an order unity $\mathbb{1}$ in E then the mappings $\pi \mapsto \pi\mathbb{1}$ from $\mathfrak{P}(E)$ into $\mathfrak{E}(E)$ and $e \mapsto \{e\}^{\perp\perp}$ from $\mathfrak{E}(E)$ into $\mathfrak{B}(E)$ are isomorphisms of Boolean algebras too.*

(2) Theorem. *Each universally complete K-space E with order unity $\mathbb{1}$ can be uniquely endowed by multiplication so as to make E into a faithful f -algebra and $\mathbb{1}$ into a ring unity. In this f -algebra each band projection $\pi \in \mathfrak{P}(E)$ is the operator of multiplication by $\pi(\mathbb{1})$.*

4.3. By a *field of reals* we mean every algebraic system that satisfies the axioms of an Archimedean ordered field (with distinct zero and unity) and enjoys the axiom of completeness. The same object can be defined as a one-dimensional *K-space*.

Recall the well-known assertion of ZFC: *There exists a field of reals \mathbb{R} that is unique up to isomorphism.*

Successively applying the transfer and maximum principles, we find an element $\mathcal{R} \in \mathbb{V}^{(\mathbb{B})}$ for which $\llbracket \mathcal{R} \text{ is a field of reals} \rrbracket = \mathbb{1}$. Moreover, if an arbitrary $\mathcal{R}' \in \mathbb{V}^{(\mathbb{B})}$ satisfies the condition $\llbracket \mathcal{R}' \text{ is a field of reals} \rrbracket = \mathbb{1}$ then $\llbracket \text{the ordered fields } \mathcal{R} \text{ and } \mathcal{R}' \text{ are isomorphic} \rrbracket = \mathbb{1}$. In other words, there exists an internal field of reals $\mathcal{R} \in \mathbb{V}^{(\mathbb{B})}$ which is unique up to isomorphism.

By the same reasons there exists an internal field of complex numbers $\mathcal{C} \in \mathbb{V}^{(\mathbb{B})}$ which is unique up to isomorphism. Moreover, $\mathbb{V}^{(\mathbb{B})} \models \mathcal{C} = \mathcal{R} \oplus i\mathcal{R}$. We call \mathcal{R} and \mathcal{C} the *internal reals* and *internal complexes* in $\mathbb{V}^{(\mathbb{B})}$.

4.4. Consider another well-known assertion of ZFC: *If \mathbb{P} is an Archimedean ordered field then there is an isomorphic embedding h of the field \mathbb{P} into \mathbb{R} such that the image $h(\mathbb{P})$ is a subfield of \mathbb{R} containing the subfield of rational numbers. In particular, $h(\mathbb{P})$ is dense in \mathbb{R} .*

Note also that $\varphi(x)$, presenting the conjunction of the axioms of an Archimedean ordered field x , is bounded; therefore, $\llbracket \varphi(\mathbb{R}^\wedge) \rrbracket = \mathbb{1}$, i.e., $\llbracket \mathbb{R}^\wedge \text{ is an Archimedean ordered field} \rrbracket = \mathbb{1}$. “Pulling” 4.2 (2) through the transfer principle, we conclude

that $\llbracket \mathbb{R}^\wedge \text{ is isomorphic to a dense subfield of } \mathcal{R} \rrbracket = 1$. We further assume that \mathbb{R}^\wedge is a dense subfield of \mathcal{R} and \mathbb{C}^\wedge is a dense subfield of \mathcal{C} . It is easy to note that the elements 0^\wedge and 1^\wedge are the zero and unity of \mathcal{R} .

Observe that the equalities $\mathcal{R} = \mathbb{R}^\wedge$ and $\mathcal{C} = \mathbb{C}^\wedge$ are not valid in general. Indeed, the axiom of completeness for \mathbb{R} is not a bounded formula and so it may thus fail for \mathbb{R}^\wedge inside $\mathbb{V}^{(\mathbb{B})}$.

4.5. Look now at the descent $\mathcal{R} \downarrow$ of the algebraic system \mathcal{R} . In other words, consider the descent of the underlying set of the system \mathcal{R} together with descended operations and order. For simplicity, we denote the operations and order in \mathcal{R} and $\mathcal{R} \downarrow$ by the same symbols $+$, \cdot , and \leq . In more detail, we introduce addition, multiplication, and order in $\mathcal{R} \downarrow$ by the formulas

$$\begin{aligned} z = x + y &\leftrightarrow \llbracket z = x + y \rrbracket = 1, \\ z = x \cdot y &\leftrightarrow \llbracket z = x \cdot y \rrbracket = 1, \\ x \leq y &\leftrightarrow \llbracket x \leq y \rrbracket = 1 \quad (x, y, z \in \mathcal{R} \downarrow). \end{aligned}$$

Also, we may introduce multiplication by the usual reals in $\mathcal{R} \downarrow$ by the rule

$$y = \lambda x \leftrightarrow \llbracket \lambda^\wedge x = y \rrbracket = 1 \quad (\lambda \in \mathbb{R}, x, y \in \mathcal{R} \downarrow).$$

One on the most fundamental results of Boolean valued analysis reads: *Each universally complete Kantorovich space is an interpretation of the reals in an appropriate Boolean valued model.* In other words, we have the following

4.6. Gordon Theorem. *Let \mathcal{R} be the reals inside $\mathbb{V}^{(\mathbb{B})}$. Then $\mathcal{R} \downarrow$, (with the descended operations and order, is a universally complete K -space with order unity 1. Moreover, there exists an isomorphism χ of \mathbb{B} onto $\mathfrak{P}(\mathcal{R} \downarrow)$ such that*

$$\chi(b)x = \chi(b)y \leftrightarrow b \leq \llbracket x = y \rrbracket, \quad \chi(b)x \leq \chi(b)y \leftrightarrow b \leq \llbracket x \leq y \rrbracket$$

for all $x, y \in \mathcal{R} \downarrow$ and $b \in \mathbb{B}$.

The converse is also true: *Each Archimedean vector lattice embeds in a Boolean valued model, becoming a vector sublattice of the reals (viewed as such over some dense subfield of the reals).*

4.7. Theorem. *Let E be an Archimedean vector lattice, let \mathcal{R} be the reals inside $\mathbb{V}^{(\mathbb{B})}$, and let j be an isomorphism of \mathbb{B} onto $\mathfrak{B}(E)$. Then there is $\mathcal{E} \in \mathbb{V}^{(\mathbb{B})}$ such that*

- (1) \mathcal{E} is a vector sublattice of \mathcal{R} over \mathbb{R}^\wedge inside $\mathbb{V}^{(\mathbb{B})}$;
- (2) $E' := \mathcal{E} \downarrow$ is a vector sublattice of $\mathcal{R} \downarrow$ invariant under every band projection $\chi(b)$ ($b \in \mathbb{B}$) and such that each set of positive pairwise disjoint elements in it has a supremum;
- (3) there is an o -continuous lattice isomorphism $\iota : E \rightarrow E'$ such that $\iota(E)$ is a coinital sublattice of $\mathcal{R} \downarrow$;
- (4) for every $b \in \mathbb{B}$ the band projection in $\mathcal{R} \downarrow$ onto $\{\iota(j(b))\}^{\perp\perp}$ coincides with $\chi(b)$.

Note also that \mathcal{E} and \mathcal{R} coincide if and only if E is Dedekind complete. Thus, each theorem about the reals within Zermelo–Fraenkel set theory has an analog in an arbitrary Kantorovich space. Translation of theorems is carried out by appropriate general functors of Boolean valued analysis. In particular, the most important structural properties of vector lattices such as the functional representation, spectral theorem, etc. are the ghosts of some properties of the reals in an appropriate Boolean valued model. More details and references are collected in [37].

4.8. The theory of vector lattices with a vast field of applications is thoroughly covered in many monographs (see [5, 2, 26, 27, 46, 59, 58, 69, 71]). The credit for finding the most important instance among ordered vector spaces, an order complete vector lattice or K -space, is due to L. V. Kantorovich. This notion appeared in Kantorovich's first article on this topic [25] where he wrote: "In this note, I define a new type of space that I call a semiordered linear space. The introduction of such a space allows us to study linear operations of one abstract class (those with values in such a space) as linear functionals."

Thus the *heuristic transfer principle* was stated for K -spaces which becomes the Ariadna thread of many subsequent studies. The depth and universality of Kantorovich's principle are explained within Boolean valued analysis.

4.9. Applications of Boolean valued models to functional analysis stem from the works by E. I. Gordon [12, 13] and G. Takeuti [65]. If \mathbb{B} in 4.6 is the algebra of μ -measurable sets modulo μ -negligible sets then $\mathcal{R}\downarrow$ is isomorphic to the universally complete K -space $L^0(\mu)$ of measurable functions. This fact (for the Lebesgue measure on an interval) was already known to D. Scott and R. Solovay (see [37]). If \mathbb{B} is a complete Boolean algebra of projections in a Hilbert space then $\mathcal{R}\downarrow$ is isomorphic to the space of selfadjoint operators $\mathfrak{A}(\mathbb{B})$. These two particular cases of Gordon's Theorem were intensively and fruitfully exploited by G. Takeuti (see [65] and the bibliography in [37]). The object $\mathcal{R}\downarrow$ for general Boolean algebras was also studied by T. Jech [21]–[23] who in fact rediscovered Gordon's Theorem. The difference is that in [24] a (complex) universally complete K -space with unity is defined by another system of axioms and is referred to as a *complete Stone algebra*. Theorem 4.7 was obtained by A. G. Kusraev [31]. A close result (in other terms) is presented in T. Jech's article [23] where some Boolean valued interpretation is revealed of the theory of linearly ordered sets. More details can be found in [37].

5. BAND PRESERVING OPERATORS

This section deals with the class of band preserving operators. Simplicity of these operators notwithstanding, the question about their order boundedness is far from trivial.

5.1. Recall that a complex K -space is the complexification $G_{\mathbb{C}} := G \oplus iG$ of a real K -space G (see [59]). A linear operator $T : G_{\mathbb{C}} \rightarrow G_{\mathbb{C}}$ is *band preserving*, or *contractive*, or a *stabilizer* if, for all $f, g \in G_{\mathbb{C}}$, from $f \perp g$ it follows that $Tf \perp g$. Disjointness in $G_{\mathbb{C}}$ is defined just as in G (see 4.1), whereas $|z| := \sup\{\operatorname{Re}(e^{i\theta}z) : 0 \leq \theta \leq \pi\}$ for $z \in G_{\mathbb{C}}$. Thus, a linear operator is band preserving if every band is its invariant subspace.

(1) Let $\operatorname{End}_N(G_{\mathbb{C}})$ stand for the set of all band preserving linear operators in $G_{\mathbb{C}}$, with $G := \mathcal{R}\downarrow$. Clearly, $\operatorname{End}_N(G_{\mathbb{C}})$ is a complex vector space. Moreover, $\operatorname{End}_N(G_{\mathbb{C}})$ becomes a faithful unitary module over the ring $G_{\mathbb{C}}$ if we define gT as $gT : x \mapsto g \cdot Tx$ for all $x \in G$. This follows from the fact that multiplication by a member of $G_{\mathbb{C}}$ is a band preserving operator and the composite of band preserving operators is band preserving too.

(2) Denote by $\operatorname{End}_{\mathbb{C}^\wedge}(\mathcal{C})$ the element of $\mathbb{V}^{(\mathbb{B})}$ representing the space of all \mathbb{C}^\wedge -linear mappings from \mathcal{C} to \mathcal{C} . Then $\operatorname{End}_{\mathbb{C}^\wedge}(\mathcal{C})$ is a vector space over \mathbb{C}^\wedge inside $\mathbb{V}^{(\mathbb{B})}$, and $\operatorname{End}_{\mathbb{C}^\wedge}(\mathcal{C})\downarrow$ is a faithful unitary module over $G_{\mathbb{C}}$.

5.2. Following [34] it is easy to prove that a linear operator T in the K -space $G_{\mathbb{C}}$ is band preserving if and only if T is extensional. Since each extensional mapping has an ascent, $T \in \text{End}_N(G_{\mathbb{C}})$ has the ascent $\tau := T\uparrow$ which is a unique internal functional from \mathcal{C} to \mathcal{C} such that $\llbracket \tau(x) = Tx \rrbracket = 1$ ($x \in G_{\mathbb{C}}$). We thus arrive at the following assertion:

The modules $\text{End}_N(G_{\mathbb{C}})$ of all linear band preserving operators in the complex K -space $G_{\mathbb{C}}$ and the descent of the internal space $\text{End}_{\mathbb{C}^\wedge}(\mathcal{C})\downarrow$ of \mathbb{C}^\wedge -linear functions in the internal complexes \mathcal{C} (considered as a vector space over \mathbb{C}^\wedge) are isomorphic by sending each band preserving operator to its ascent.

By Gordon's Theorem this assertion means that the problem of finding a band preserving operator in $G_{\mathbb{C}}$ amounts to solving (for $\tau : \mathcal{C} \rightarrow \mathcal{C}$) inside $\mathbb{V}^{(\mathbb{B})}$ the *Cauchy functional equation*: $\tau(x+y) = \tau(x) + \tau(y)$ ($x, y \in \mathcal{C}$) under the subsidiary condition $\tau(\lambda x) = \lambda \tau(x)$ ($x \in \mathcal{C}, \lambda \in \mathbb{C}^\wedge$).

As another subsidiary condition we may consider the *Leibniz rule* $\tau(xy) = \tau(x)y + x\tau(y)$ (in which case τ is called a \mathbb{C}^\wedge -*derivation*) or multiplicativity $\tau(xy) = \tau(x)\tau(y)$. These situations are addressed in 5.5.

5.3. An element $g \in G^+$ is *locally constant* with respect to $f \in G^+$ if $g = \bigvee_{\xi \in \Xi} \lambda_\xi \pi_\xi f$ for some numeric family $(\lambda_\xi)_{\xi \in \Xi}$ and a family $(\pi_\xi)_{\xi \in \Xi}$ of pairwise disjoint band projections. A universally complete K -space $G_{\mathbb{C}}$ is called *locally one-dimensional* if all elements of G^+ are locally constant with respect to some order unity of G (and hence each of them). Clearly, a K -space $G_{\mathbb{C}}$ is locally one-dimensional if each $g \in G_{\mathbb{C}}$ may be presented as $g = o\text{-}\sum_{\xi \in \Xi} \lambda_\xi \pi_\xi 1f$ with some family $(\lambda_\xi)_{\xi \in \Xi} \subset \mathbb{C}$ and partition of unity $(\pi_\xi)_{\xi \in \Xi} \subset \mathfrak{P}(G)$.

Theorem *Let $G_{\mathbb{C}}$ be a universally complete K -space. Every band preserving linear operator in $G_{\mathbb{C}}$ is order bounded if and only if $G_{\mathbb{C}}$ is locally one-dimensional.*

5.4. A σ -complete Boolean algebra \mathbb{B} is called σ -*distributive* if

$$\bigvee_{n \in \mathbb{N}} \bigwedge_{m \in \mathbb{N}} b_{n,m} = \bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \bigvee_{n \in \mathbb{N}} b_{n, \varphi(n)}.$$

for every double sequence $(b_{n,m})_{n,m \in \mathbb{N}}$ in \mathbb{B} .

An equivalent definition can be given in terms of partitions of unity. From any two partitions of unity in an arbitrary Boolean algebra one can refine a partition of unity by taking infimum of any pair of members of the partitions. The same is true for a finite set of partitions of unity. A σ -complete Boolean algebra \mathbb{B} is called σ -distributive if from every sequence of countable partitions of unity in \mathbb{B} , it is possible to refine a (possibly, uncountable) partition of unity.

Other equivalent definitions are collected in [61]. As an example of a σ -distributive Boolean algebra we may take a complete atomic Boolean algebra, i.e., the boolean of a nonempty set. It is worth observing that there are nonatomic σ -distributive complete Boolean algebras (see [33, 5.1.8]).

5.5. We now address the problem which is often referred to in the literature as *Wickstead's problem*: Characterize the universally complete vector lattices in which every band preserving linear operator is order bounded. We restrict exposition to the case of complex vector lattices.

According to 5.2, Boolean valued analysis reduces Wickstead's problem to that of order boundedness of the endomorphisms of the field \mathcal{C} viewed as a vector space and algebra over \mathbb{C}^\wedge . It is important that the standard name of the external complexes is an algebraically closed field inside $\mathbb{V}^{(\mathbb{B})}$:

The field \mathbb{C}^\wedge is algebraically closed in \mathcal{C} inside $\mathbb{V}^{(\mathbb{B})}$. In particular, if $\mathbb{C}^\wedge \neq \mathcal{C}$ then

$$\mathbb{V}^{(\mathbb{B})} \models \text{“}\mathcal{C} \text{ — transcendental extension of the field” } \mathbb{C}^\wedge.$$

We so arrived at an internal Cauchy type functional equation: Find an additive function in the internal complexes that is \mathbb{P} -homogeneous for some algebraically closed dense subfield \mathbb{P} . (If \mathbb{P} coincides with the field of rationals then we obtain exactly the Cauchy functional equation inside the Boolean valued universe.) The corresponding scalar result reads as follows.

5.6. Theorem. *Let \mathbb{P} be an algebraically closed and (topologically) dense subfield of the field of complexes \mathbb{C} . The following are equivalent:*

- (1) $\mathbb{P} = \mathbb{C}$;
- (2) every \mathbb{P} -linear function on \mathbb{C} is order bounded;
- (3) there are no nontrivial \mathbb{P} -derivations on \mathbb{C} ;
- (4) each \mathbb{P} -linear endomorphism on \mathbb{C} is the zero or identity function;
- (5) there is no \mathbb{P} -linear automorphism on \mathbb{C} other than the identity.

The equivalence (1) \leftrightarrow (2) is checked by using a Hamel basis of the vector space \mathbb{C} over \mathbb{P} . The remaining equivalences rest on replacing a Hamel basis with a transcendence basis (for details see [35]).

Recall that a linear operator $D : G_{\mathbb{C}} \rightarrow G_{\mathbb{C}}$ is a \mathbb{C} -derivation if it obeys the Leibnitz rule $D(fg) = D(f)g + fD(g)$ for all $f, g \in G_{\mathbb{C}}$. It can be easily checked that every \mathbb{C} -derivation is band preserving.

Interpreting Theorem 5.5 in $\mathbb{V}^{(\mathbb{B})}$, we arrive at following two results.

5.7. Theorem. *If \mathbb{B} is a complete Boolean algebra then the following are equivalent:*

- (1) $\mathcal{C} = \mathbb{C}^\wedge$ inside $\mathbb{V}^{(\mathbb{B})}$;
- (2) every band preserving linear operator is order bounded in the complex vector lattice $\mathcal{C} \downarrow$;
- (3) \mathbb{B} is σ -distributive.

5.8. Theorem. *If \mathbb{B} is a complete Boolean algebra then the following are equivalent:*

- (1) $\mathcal{C} = \mathbb{C}^\wedge$ inside $\mathbb{V}^{(\mathbb{B})}$;
- (2) there is no nontrivial \mathbb{C} -derivation in the complex f -algebra $\mathcal{C} \downarrow$;
- (3) each band preserving endomorphism is a band projection in $\mathcal{C} \downarrow$;
- (4) there is no band preserving automorphism other than the identity in $\mathcal{C} \downarrow$.
- (5) \mathbb{B} is σ -distributive.

5.9. The above problem was posed by A. W. Wickstead in [70]. The first example of an unbounded band preserving linear operator was suggested by Yu. A. Abramovich, A. I. Veksler, and A. V. Koldunov in [8, 7]. Theorem 5.3 combines a result of Yu. A. Abramovich, A. I. Veksler, and A. V. Koldunov [8, Theorem 2.1] and that of P. T. N. McPolin and A. W. Wickstead [48, Theorem 3.2]. Theorem 4,7 was obtained by A. E. Gutman [16]; he also found an example of a purely nonatomic locally one-dimensional Dedekind complete vector lattice (see [17]). Theorem 5.8 belong to A. G. Kusraev [35].

6. ORDER BOUNDED OPERATORS

A linear functional on a vector space is determined up to a scalar from its zero hyperplane. In contrast, a linear operator is recovered from its kernel up to a simple multiplier on a rather special occasion. Fortunately, Boolean valued analysis prompts us that some operator analog of the functional case is valid for each operator with target a Kantorovich space, a Dedekind complete vector lattice. We now proceed along the lines of this rather promising approach.

6.1. Let E be a vector lattice, and let F be a K -space with base a complete Boolean algebra \mathbb{B} . By 4.2, we may assume that F is a nonzero space embedded as an order dense ideal in the universally complete Kantorovich space $\mathcal{R}\downarrow$ which is the descent of the reals \mathcal{R} inside the separated Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$ over \mathbb{B} .

An operator T is F -discrete if $[0, T] = [0, I_F] \circ T$; i.e., for all $0 \leq S \leq T$ there is some $0 \leq \alpha \leq I_F$ satisfying $S = \alpha \circ T$. Let $L_a^\sim(E, F)$ be the band in $L^\sim(E, F)$ spanned by F -discrete operators and $L_d^\sim(E, F) := L_a^\sim(E, F)^\perp$. By analogy we define $(E^{\wedge\sim})_a$ and $(E^{\wedge\sim})_d$. The members of $L_d^\sim(E, F)$ are usually called F -diffuse.

6.2. As usual, we let E^\wedge stand for the standard name of E in $\mathbb{V}^{(\mathbb{B})}$. Clearly, E^\wedge is a vector lattice over \mathbb{R}^\wedge inside $\mathbb{V}^{(\mathbb{B})}$. Denote by $\tau := T\uparrow$ the ascent of T to $\mathbb{V}^{(\mathbb{B})}$. Clearly, τ acts from E^\wedge to the ascent $F\uparrow = \mathcal{R}$ of F inside the Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$. Therefore, $\tau(x^\wedge) = Tx$ inside $\mathbb{V}^{(\mathbb{B})}$ for all $x \in E$, which means in terms of truth values that $\llbracket \tau : E^\wedge \rightarrow \mathcal{R} \rrbracket = 1$ and $(\forall x \in E) \llbracket \tau(x^\wedge) = Tx \rrbracket = 1$.

Let $E^{\wedge\sim}$ stand for the space of all order bounded \mathbb{R}^\wedge -linear functionals from E^\wedge to \mathcal{R} . Clearly, $E^{\wedge\sim} := L^\sim(E^\wedge, \mathcal{R})$ is a K -space inside $\mathbb{V}^{(\mathbb{B})}$. The descent $E^{\wedge\sim}\downarrow$ of $E^{\wedge\sim}$ is a K -space. Given $S, T \in L^\sim(E, F)$, put $\tau := T\uparrow$ and $\sigma := S\uparrow$.

6.3. Theorem. *For each $T \in L^\sim(E, F)$ the ascent $T\uparrow$ of T is an order bounded \mathbb{R}^\wedge -linear functional on E^\wedge inside $\mathbb{V}^{(\mathbb{B})}$; i.e., $\llbracket T\uparrow \in E^{\wedge\sim} \rrbracket = 1$. The mapping $T \mapsto T\uparrow$ is a lattice isomorphism of $L^\sim(E, F)$ and $E^{\wedge\sim}\downarrow$. In particular, the following hold:*

- (1) $T \geq 0 \leftrightarrow \llbracket \tau \geq 0 \rrbracket = 1$;
- (2) S is a fragment of $T \leftrightarrow \llbracket \sigma \text{ is a fragment of } \tau \rrbracket = 1$;
- (3) T is a lattice homomorphism if and only if so is τ inside $\mathbb{V}^{(\mathbb{B})}$;
- (4) T is F -diffuse $\leftrightarrow \llbracket \tau \text{ is diffuse} \rrbracket = 1$;
- (5) $T \in L_a^\sim(E, F) \leftrightarrow \llbracket \tau \in (E^{\wedge\sim})_a \rrbracket = 1$;
- (6) $T \in L_d^\sim(E, F) \leftrightarrow \llbracket \tau \in (E^{\wedge\sim})_d \rrbracket = 1$.

Thus, the ascent and descent operations implement a lattice isomorphism of the vector lattice of all linear order bounded operators from E to F and the descent of the internal space of all linear order bounded functionals in the standard name of E . This Boolean valued representation does not preserve order continuity and is not suitable for the study of order continuous operators. But it may reduce some problems on general order bounded and positive operators to those on functionals and provide a rather promising approach.

Consider an instance of this approach. A linear functional on a vector space is determined up to a scalar from its zero hyperplane. In contrast, a linear operator is recovered from its kernel up to a simple multiplier on a rather special occasion. Fortunately, Boolean valued analysis prompts us that some operator analog of the functional case is valid for each operator with target a Kantorovich space.

More precisely, since τ , the ascent of an order bounded operator T , is defined up to a scalar from $\ker(\tau)$, we infer the following analog of the Sard Theorem.

6.4. Theorem. *Let S and T be linear operators from E to F . Then $\ker(bS) \supset \ker(bT)$ for all $b \in \mathbb{B}$ if and only if there is an orthomorphism α of F such that $S = \alpha T$.*

We see that a linear operator T is, in a sense, determined up to an orthomorphism from the family of the kernels of the *strata* bT of T . This remark opens a possibility of studying some properties of T in terms of the kernels of the strata of T .

6.5. Theorem. *An order bounded operator T from E to F may be presented as the difference of some lattice homomorphisms if and only if the kernel of each stratum bT of T is a vector sublattice of E for all $b \in \mathbb{B}$.*

Straightforward calculations of truth values show that $T_+ \uparrow = \tau_+$ and $T_- \uparrow = \tau_-$ inside $\mathbb{V}^{(\mathbb{B})}$. Moreover, $\llbracket \ker(\tau) \rrbracket$ is a vector sublattice of $E^\wedge = 1$ whenever so are $\ker(bT)$ for all $b \in \mathbb{B}$. Since the ascent of a sum is the sum of the ascents of the summands, we reduce the proof of Theorem 6.5 to the case of the functionals on using 6.3 (3).

6.6. Recall that a subspace H of a vector lattice E is a *G-space* or *Grothendieck subspace* (cp. [15, 44]) provided that H enjoys the following property:

$$(\forall x, y \in H) (x \vee y \vee 0 + x \wedge y \wedge 0 \in H).$$

By simple calculations of truth values we infer that $\llbracket \ker(\tau) \rrbracket$ is a Grothendieck subspace of $E^\wedge = 1$ if and only if the kernel of each stratum bT is a Grothendieck subspace of E . We may now assert that the following appears as a result of “descending” its scalar analog.

6.7. Theorem. *The modulus of an order bounded operator $T : E \rightarrow F$ is the sum of some pair of lattice homomorphisms if and only if the kernel of each stratum bT of T with $b \in \mathbb{B}$ is a Grothendieck subspace of the ambient vector lattice E .*

To prove the relevant scalar versions of Theorems 6.5 and 6.7, we use one of the formulas of subdifferential calculus (cp. [36]):

6.8. Decomposition Theorem. *Assume that H_1, \dots, H_N are cones in a vector lattice E . Assume further that f and g are positive functionals on E . The inequality $f(h_1 \vee \dots \vee h_N) \geq g(h_1 \vee \dots \vee h_N)$ holds for all $h_k \in H_k$ ($k := 1, \dots, N$) if and only if to each decomposition of g into a sum of N positive terms $g = g_1 + \dots + g_N$ there is a decomposition of f into a sum of N positive terms $f = f_1 + \dots + f_N$ such that $f_k(h_k) \geq g_k(h_k)$ ($h_k \in H_k$; $k := 1, \dots, N$).*

6.9. Theorems 6.5 and 6.7 were obtained by S. S. Kutateladze in [42, 43]. Theorem 6.8 appeared in this form in [39]. Note that the sums of lattice homomorphisms were first described by S. J. Bernau, C. B. Huijsmans, and B. de Pagter in terms of n -disjoint operators in [10]. A survey of some conceptually close results on n -disjoint operators is given in [33].

7. FRAGMENTS OF POSITIVE OPERATORS

In this section the tools for generating fragments of positive operators and representation of principal band projections are described. As above, we use the general method of ascending into a Boolean-valued universe and descending the corresponding results for functionals.

7.1. A set of band projections \mathcal{P} in the K -space $L^\sim(E, F)$ *generates* the fragments of a positive operator $T \in L^\sim(E, F)_+$ provided that $Tx^+ = \sup\{(pT)x : p \in \mathcal{P}\}$. If the latter is true for all $T \in L^\sim(E, F)_+$ and $x \in E$ then \mathcal{P} is said

to be a *generating set*. As an easy example we cite the following. To each band projection $\pi \in \mathfrak{P}(E)$ assign the band projection $\hat{\pi}T \mapsto T \circ \pi$ acting in $L^\sim(E, F)$ and denoted by \mathcal{P}° the set of all such band projections. It can be easily checked that if a vector lattice E has projection property then \mathcal{P}° is a generating set of projections in $L^\sim(E, F)$.

Put $\mathcal{P}^\pi := \{\pi_e : e \in E_+\}$ where π_e is defined as follows:

$$\begin{aligned}\pi_e Tx &= \sup_n T(ne \wedge x) \quad (x \in E^+, T \in L^+(E, F)), \\ \pi_e Tx &= \pi_e Tx^+ - \pi_e Tx^- \quad (x \in E, T \in L^+(E, F)), \\ \pi_e T &= \pi_e T^+ - \pi_e T^- \quad (T \in L^\sim(E, F)).\end{aligned}$$

Then \mathcal{P}^π is a generating set of projections in $L^\sim(E, F)$.

7.2. According to 6.3 the mapping $T \in L^\sim(E, F) \mapsto T\uparrow \in E^{\wedge\sim}\downarrow$ implements an isomorphism between the structures of $L^\sim(E, F)$ and $E^{\wedge\sim}\downarrow$. Therefore, T is a fragment of S or T is in $S^{\perp\perp}$ if and only if $T\uparrow$ is a fragment of $S\uparrow$ or $T\uparrow$ is in $\{S\uparrow\}^{\perp\perp}$ inside $\mathbb{V}^{(\mathbb{B})}$.

The mapping $T\uparrow \mapsto (pT)\uparrow$ ($T \in L^\sim(E, F)$) is extensional for $p \in \mathcal{P}$. By analogy, the ascent $p\uparrow$ is defined to be the band projection in $E^{\wedge\sim}$ inside $\mathbb{V}^{(\mathbb{B})}$ acting by the rule $p\uparrow T\uparrow = (pT)\uparrow$ for $T \in L^\sim(E, F)$.

Now, consider the ascent $\mathcal{P}\uparrow$ defined as $\mathcal{P}\uparrow := \{p\uparrow \mid p \in \mathcal{P}\}\uparrow$. Obviously, \mathcal{P} generates the fragments of T if and only if $\mathcal{P}\uparrow$ generates the fragments of $T\uparrow$ inside $\mathbb{V}^{(\mathbb{B})}$.

Given a set A in a K -space, we denote by A^\vee the union of A and the suprema of all nonempty finite subsets of A . The symbol $A^{(\uparrow)}$ denotes the result of adjoining to A the suprema of all increasing nonempty nets in A . The symbols $A^{(\uparrow\downarrow)}$ and $A^{(\uparrow\uparrow)}$ are interpreted in a natural way.

7.3. Let \mathbb{P} be a dense subfield of \mathbb{R} and E be a vector lattice over \mathbb{P} . Denote by $E^\sim := L^\sim(E, \mathbb{R})$ the vector lattice of \mathbb{P} -linear functionals in E . Fix some set \mathcal{P} of band projections and the corresponding set $\mathcal{P}(f) := \{pf : p \in \mathcal{P}\}$ of the fragments of a positive functional $f \in E^\sim$.

Theorem. For positive functionals $f, g \in E^\sim$ the following are true:

- (1) \mathcal{P} generates the fragments of f if and only if $\mathcal{P}(f)^{\vee(\uparrow\uparrow)} = \mathfrak{C}(f)$;
- (2) if \mathcal{P} is generating then $g \in \{f\}^{\perp\perp}$ if and only if for any $x \in E_+$ and $0 < \varepsilon \in \mathbb{R}$ there exists $0 < \delta \in \mathbb{R}$ such that $pf(x) \leq \delta$ implies $pg(x) \leq \varepsilon$ for every $p \in \mathcal{P}$;
- (3) if \mathcal{P} is generating then for the principal band projection π_f onto $\{f\}^{\perp\perp}$ the representations hold:

$$\pi_f g(x) = \sup_{\varepsilon > 0} \inf \{pg(x) : p^\perp f(x) \leq \varepsilon, p \in \mathcal{P}\}.$$

7.4. Theorem. A set \mathcal{P} of band projections in $L^\sim(E, F)$ generates the fragments of $T \in L^\sim(E, F)_+$ if and only if $\mathcal{P}(T)^{\vee(\uparrow\uparrow)} = \mathfrak{C}(T)$.

7.5. Theorem. If \mathcal{P} is a generating set of projections in $L^\sim(E, F)$ then for positive operators $S, T \in L^\sim(E, F)$ the relation $T \in \{S\}^{\perp\perp}$ holds if and only if for every $e \in E_+$ and $0 < \varepsilon \in \mathbb{R}$ there exists $0 < \delta \in F$, $[Se] \leq [\delta]$, $\delta \leq Se$, such that $\pi p Se \leq \delta$ implies $\pi p Te \leq \varepsilon Te$ for all $\pi \in \mathfrak{P}(F)$ and $p \in \mathcal{P}$.

7.6. Theorem. Let E be an arbitrary vector lattice, F be a K -space, and \mathcal{P} be a generating set of projections. Then for the band projection T_S of T onto $\{S\}^{\perp\perp}$

the representations are valid:

$$(T - T_S)e = \inf_{0 < \varepsilon \in \mathbb{R}} \sup\{\pi p T e : \pi \in \mathfrak{P}(F), p \in \mathscr{P}, \pi p S e \leq \varepsilon S e\},$$

$$(T_S)e = \sup_{0 < \varepsilon \in \mathbb{R}} \inf\{(\pi p)^\perp T e : \pi \in \mathfrak{P}(F), p \in \mathscr{P}, \pi p S e \leq \varepsilon S e\}.$$

7.7. The concept of a generating set of projections as well as Theorem 7.4 belongs to S. S. Kutateladze [40]. In 7.4 every fragment of a positive operator is obtained from its simpler fragments by up and down procedures. Similar assertions are often referred to as *up-down theorems*. The first up-down theorem was established by B. de Pagter [56] (also see [1], [2]). However, it involved two essential constraints: F should admit a total set of σ -continuous functionals, and E must be order complete (or at least possess the principal projection property). The first constraint was eliminated in [38] and the second, in [4]. Of course, a few up-down theorems can be deduced from 7.4 by specifying generating sets (see [33] for details). Theorems 7.5 and 7.6 are improved versions of the corresponding results of [40].

8. BOOLEAN VALUED BANACH SPACES

In this section we discuss the transfer principle of Boolean valued analysis in regard to lattice-normed spaces. It turns out that the interpretation of a Banach space inside an arbitrary Boolean valued model is a Banach–Kantorovich space. Conversely, the universal completion of each lattice-normed space becomes a Banach space on ascending in a suitable Boolean valued model. This opens up an opportunity to transfer the available theorems on Banach spaces to analogous results on lattice-normed spaces by the technique of Boolean valued analysis.

8.1. Consider a vector space X and a real vector lattice E . Note that all vector lattices under consideration are assumed Archimedean. An E -valued norm is a mapping $|\cdot| : X \rightarrow E_+$ such that

- (1) $|x| = 0 \iff x = 0 \quad (x \in X)$;
- (2) $|\lambda x| = |\lambda| |x| \quad (\lambda \in \mathbb{R}, x \in X)$;
- (3) $|x + y| \leq |x| + |y| \quad (x, y \in X)$.

A vector norm is *decomposable* if

- (4) for all $e_1, e_2 \in E_+$ and $x \in X$, from $|x| = e_1 + e_2$ it follows that there exist $x_1, x_2 \in X$ such that $x = x_1 + x_2$ and $|x_k| = e_k \quad (k := 1, 2)$.

If (4) is valid only for disjoint $e_1, e_2 \in E_+$ then the norm is *d-decomposable*. A triple $(X, |\cdot|, E)$ as well as briefer versions is a *lattice-normed space* over E whenever $|\cdot|$ is an E -valued norm on X .

8.2. By a *Boolean algebra of projections* in a vector space X we mean a set \mathscr{B} of commuting idempotent linear operators in X . Moreover, the Boolean operations have the following form:

$$\pi \wedge \rho := \pi \circ \rho = \rho \circ \pi, \quad \pi \vee \rho = \pi + \rho - \pi \circ \rho,$$

$$\pi^* = I_x - \pi \quad (\pi, \rho \in \mathscr{B}),$$

and the zero and identity operators in X serve as the zero and unity of the Boolean algebra \mathscr{B} .

Suppose that E is a vector lattice with the projection property and $E = |X|^{\perp\perp} := \{|x| : x \in X\}^{\perp\perp}$. If (X, E) is a *d-decomposable lattice-normed space* then there

exists a complete Boolean algebra \mathcal{B} of band projections in X and an isomorphism h from $\mathfrak{P}(E)$ onto \mathcal{B} such that

$$b|x| = |h(b)x| \quad (b \in \mathfrak{P}(E), x \in X).$$

We identify the Boolean algebras $\mathfrak{P}(E)$ and \mathcal{B} and write $\pi|x| = |\pi x|$ for all $x \in X$ and $\pi \in \mathfrak{P}(E)$.

8.3. A net $(x_\alpha)_{\alpha \in A}$ in X is *bo-convergent* to $x \in X$ (in symbols: $x = \text{bo-lim } x_\alpha$) if $(|x - x_\alpha|)_{\alpha \in A}$ is *o-convergent* to zero. A lattice-normed space X is *bo-complete* if each net $(x_\alpha)_{\alpha \in A}$ is *bo-convergent* to some element of X provided that $(|x_\alpha - x_\beta|)_{(\alpha, \beta) \in A \times A}$ is *o-convergent* to zero. A decomposable *bo-complete* lattice-normed space $(X, |\cdot|, E)$ is called a *Banach–Kantorovich space*. If E is a universally complete Kantorovich space then X is also referred to as universally complete. By a *universal completion* of a lattice-normed space (X, E) we mean a universally complete Banach–Kantorovich space $(Y, m(E))$ together with a linear isometry $\iota : X \rightarrow Y$ such that each universally complete *bo-complete* subspace of $(Y, m(E))$ containing $\iota(X)$ coincides with Y . Here $m(E)$ is a universal completion of E .

8.4. Theorem. Let $(\mathcal{X}, \|\cdot\|)$ be a Banach space inside $\mathbb{V}^{(\mathbb{B})}$. Put $X := \mathcal{X} \downarrow$ and $|\cdot| := \|\cdot\| \downarrow (\cdot)$. Then $(X, |\cdot|, \mathcal{X} \downarrow)$ is a universally complete Banach–Kantorovich space. Moreover, X can be endowed with the structure of a faithful unitary module over the ring $\Lambda := \mathcal{C} \downarrow$ so that $|ax| = |a||x|$ and $b \leq [x = 0] \leftrightarrow \chi(b)x = 0$ for all $a \in \mathcal{C} \downarrow$, $x \in X$, and $b \in \mathbb{B}$, where χ is an isomorphism of \mathbb{B} onto $\mathfrak{P}(X)$.

8.5. Theorem. To each lattice-normed space $(X, |\cdot|)$, there exists a unique Banach space (up to a linear isometry) \mathcal{X} inside $\mathbb{V}^{(\mathbb{B})}$, with $\mathbb{B} \simeq \mathfrak{B}(|X|^{\perp\perp})$, such that the descent $\mathcal{X} \downarrow$ of \mathcal{X} is a universal completion of X .

As in 4.1, we call $x \in X$ and $y \in Y$ *disjoint* and write $x \perp y$ whenever $|x| \wedge |y| = 0$. Let X and Y be Banach–Kantorovich spaces over some K -space G .

An operator T is *band preserving* if $x \perp y$ implies $Tx \perp y$ for all $x \in X$ and $y \in Y$. Denote by $\mathcal{L}_G(X, Y)$ the space of all band preserving operators $T : X \rightarrow Y$ that send all norm-*o*-bounded sets into norm-*o*-bounded sets.

8.6. Theorem. Let \mathcal{X} and \mathcal{Y} be Boolean valued representations for Banach–Kantorovich spaces X and Y normed by some universally complete K -space $G := \mathcal{G} \downarrow$. Let $\mathcal{L}^{\mathbb{B}}(\mathcal{X}, \mathcal{Y})$ be the space of bounded linear operators from \mathcal{X} into \mathcal{Y} inside $\mathbb{V}^{(\mathbb{B})}$, where $\mathbb{B} := \mathfrak{B}(G)$. The descent and ascent mappings (for operators) implement linear isometries between the lattice-normed spaces $\mathcal{L}_G(X, Y)$ and $\mathcal{L}^{\mathbb{B}}(\mathcal{X}, \mathcal{Y}) \downarrow$.

8.7. The concept of lattice-normed space was suggested by L. V. Kantorovich in 1936 [25]. It is worth stressing that [25] is the first article with the unusual decomposability axiom for an abstract norm. Paradoxically, this axiom was often omitted as inessential in the further papers by other authors. The profound importance of 8.1 (4) was revealed by Boolean valued analysis. The connection between the decomposability and existence of a Boolean algebra of projections in a lattice-normed space was discovered in [30, 32]. The theory of lattice-normed spaces and dominated operators is set forth in [33]. As regards the Boolean valued approach, see [37].

9. BOOLEAN VALUED ORDER CONTINUOUS FUNCTIONALS

We now address the class of \mathcal{o} -continuous order bounded operators that turn into \mathcal{o} -continuous functionals on ascending to a suitable Boolean valued model.

9.1. Assume that a lattice-normed space X is simultaneously a vector lattice. The norm $|\cdot| : X \rightarrow E_+$ of X is *monotone* if from $|x| \leq |y|$ it follows that $|x| \leq |y|$ ($x, y \in X$). In this event, X is a *lattice-normed vector lattice*. Moreover, if X is a Banach–Kantorovich space then X is called a *Banach–Kantorovich lattice*.

We say that the norm $|\cdot|$ in X is *additive* if $|x + y| = |x| + |y|$ for all $x, y \in X_+$; it is *order semicontinuous* or *\mathcal{o} -semicontinuous* for short if $\sup |x_\alpha| = |\sup x_\alpha|$ for each increasing net $(x_\alpha) \subset X$ with the least upper bound $x \in X$; and it is *order continuous* or *\mathcal{o} -continuous* if $\inf |x_\alpha| = 0$ for every decreasing net $(x_\alpha) \subset X$ with $\inf_\alpha x_\alpha = 0$.

The Boolean valued interpretation of Banach–Kantorovich lattices proceeds along the lines of the previous section.

9.2. Theorem. Let $(X, |\cdot|)$ be a Banach–Kantorovich space and let $(\mathcal{X}, \|\cdot\|) \in \mathbb{V}^{(\mathbb{B})}$ stand for its Boolean valued realization. Then

(1) X is a Banach–Kantorovich lattice if and only if \mathcal{X} is a Banach lattice inside $\mathbb{V}^{(\mathbb{B})}$;

(2) X is an order complete Banach–Kantorovich lattice if and only if \mathcal{X} is an order complete Banach lattice inside $\mathbb{V}^{(\mathbb{B})}$;

(3) the norm $|\cdot|$ is \mathcal{o} -continuous (order semicontinuous, monotone complete, or additive) if and only if the norm $\|\cdot\|$ is \mathcal{o} -continuous (order semicontinuous, monotone complete, or additive) inside $\mathbb{V}^{(\mathbb{B})}$.

9.3. Let E be a vector lattice, let F be some K -space, and let T be a positive operator from E to F .

Say that T possesses the *Maharam property* if, for all $x \in E_+$ and $0 \leq f \leq Tx \in F_+$, there is some $0 \leq e \leq x$ satisfying $f = Te$. An \mathcal{o} -continuous positive operator with the Maharam property is a *Maharam operator*.

Observe that $T \in L(E, F)_+$ possesses the Maharam property if and only if the equality $T([0, x]) = [0, Tx]$ holds for all $x \in E_+$. Thus, a Maharam operator is exactly an \mathcal{o} -continuous order-interval preserving positive operator.

Let T be an essentially positive operator from E to F enjoying the Maharam property. Put $|e| := T(|x|)$ ($e \in E$). Then $(E, |\cdot|)$ is a disjointly decomposable lattice-normed space over F .

Put $F_T := \{T(|x|) : x \in E\}^{\perp\perp}$, and let $\mathcal{D}_m(T)$ stand for the greatest order dense ideal of the universal completion $m(E)$ of E among those to which T can be extended by \mathcal{o} -continuity. In other words, $z \in \mathcal{D}_m(T)$ if and only if $z \in m(E)$ and the set $\{T(x) : x \in E, 0 \leq x \leq |z|\}$ is bounded in F . In this event there exists a minimal extension of T to $\mathcal{D}_m(T)$ presenting an \mathcal{o} -continuous positive operator.

Let E and F be some K -spaces, and let $T : E \rightarrow F$ be a Maharam operator. Put $X := \mathcal{D}_m(T)$ and $|x| := \Phi(|x|)$ ($x \in X$), where Φ is an \mathcal{o} -continuous extension of T to X . Then $(X, |\cdot|)$ is a Banach–Kantorovich lattice whose norm is \mathcal{o} -continuous and additive.

9.4. Theorem. Let X be an arbitrary K -space and let E be a universally complete K -space $\mathcal{R}\downarrow$. Assume that $\Phi : X \rightarrow E$ is a Maharam operator such that $X = X_\Phi = \mathcal{D}_m(\Phi)$ and $E = E_\Phi$. Then there are elements \mathcal{X} and φ in $\mathbb{V}^{(\mathbb{B})}$ satisfying

- (1) $\llbracket \mathcal{X} \text{ is a } K\text{-space, } \varphi : \mathcal{X} \rightarrow \mathcal{R} \text{ is a positive } o\text{-continuous functional, and } \mathcal{X} = \mathcal{X}_\varphi = \mathcal{D}_m(\varphi) \rrbracket = 1$;
- (2) if $X' := \mathcal{X} \downarrow$ and $\Phi' = \varphi \downarrow$ then X' is a K -space and $\Phi' : X' \rightarrow E$ is a Maharam operator;
- (3) there is a linear and lattice isomorphism h from X onto X' such that $\Phi = \Phi' \circ h$;
- (4) for a linear operator Ψ , the containment $\Psi \in \{\Phi\}^{\perp\perp}$ is true if and only if there is $\psi \in \mathbb{V}^{(\mathbb{B})}$ such that $\psi \in \{\varphi\}^{\perp\perp}$ inside $\mathbb{V}^{(\mathbb{B})}$ and $\Psi = (\psi \downarrow) \circ h$.

Theorem 9.4 enables us to claim that each fact about o -continuous positive linear functionals in K -spaces has a parallel version for Maharam operators which can be revealed by using 9.4. For instance, we state the abstract

9.5. Radon–Nikodým Theorem. *Let E and F be K -spaces. Assume further that S and T are o -continuous positive operators from E to F , with T enjoying the Maharam property. Then the following are equivalent:*

- (1) $S \in \{T\}^{\perp\perp}$;
- (2) $Sx \in \{Tx\}^{\perp\perp}$ for all $x \in E_+$;
- (3) there is an extended orthomorphism $0 \leq \rho \in \text{Orth}^\infty(E)$ satisfying $Sx = T(\rho x)$ for all $x \in E$ such that $\rho x \in E$;
- (4) there is a sequence of orthomorphisms $(\rho_n) \subset \text{Orth}(E)$ such that $Sx = \sup_n T(\rho_n x)$ for all $x \in E$.

9.6. A brief description for Maharam’s approach to studying positive operators in the spaces of measurable functions and the main results in this area are collected in [47]. W. A. J. Luxemburg and A. R. Schep [45] extended a portion of Maharam’s theory on the Radon–Nikodým Theorem to the case of positive operators in vector lattices.

Theorem 9.2 and 9.4 were obtained by A. G. Kusraev [29] and Theorem 9.5, by W. A. J. Luxemburg and A. R. Schep [45]. About various applications of the above results on Maharam operators and some extension of this theory to sublinear and convex operators see [32, 33, 36, 37].

10. SPACES WITH MIXED NORM

The definitions of various objects of functional analysis rest often on some blending of the norm and order properties. Among these are listed the spaces with mixed norm and the classes of linear operators between them.

10.1. If (X, E) is a lattice-normed space whose *norm lattice* E is a Banach lattice. Since, by definition, $\llbracket x \rrbracket \in E$ for $x \in X$, we may introduce the *mixed norm* on X by the formula

$$\|x\| := \|\llbracket x \rrbracket\| \quad (x \in X).$$

In this situation, the normed space $(X, \|\cdot\|)$ is called a *space with mixed norm*. A *Banach space with mixed norm* is a pair (X, E) with E a Banach lattice and X a br -complete lattice-normed space with E -valued norm. The following proposition justifies this definition.

Let E be a Banach lattice. Then $(X, \|\cdot\|)$ is a Banach space if and only if the lattice-normed space (X, E) is relatively uniformly complete.

10.2. Let Λ be the *bounded part* of the universally complete K -space $\mathcal{R} \downarrow$, i.e. Λ is the order-dense ideal in $\mathcal{R} \downarrow$ generated by the order unity $1 := 1^\wedge \in \mathcal{R} \downarrow$. Take

a Banach space \mathcal{X} inside $\mathbb{V}^{(\mathbb{B})}$. Put

$$\begin{aligned}\mathcal{X} \Downarrow &:= \{x \in \mathcal{X} \downarrow : \llbracket x \rrbracket \in \Lambda\}, \\ \|x\| &:= \|\llbracket x \rrbracket\|_\infty := \inf\{0 < \lambda \in \mathbb{R} : \llbracket x \rrbracket \leq \lambda \mathbf{1}\}.\end{aligned}$$

Then $\mathcal{X} \Downarrow$ is a Banach–Kantorovich space called the *bounded descent* of \mathcal{X} . Since Λ is an order complete AM -space with unity, $\mathcal{X} \Downarrow$ is a Banach space with mixed norm over Λ .

Thus, we came to the following natural question: Which Banach spaces are linearly isometric to the bounded descents of internal Banach spaces? The answer is given in terms of \mathbb{B} -cyclic Banach spaces.

10.3. Let X be a normed space. Suppose that $\mathcal{L}(X)$ has a complete Boolean algebra of norm one projections \mathcal{B} which is isomorphic to \mathbb{B} . In this event we will identify the Boolean algebras \mathcal{B} and \mathbb{B} , writing $\mathbb{B} \subset \mathcal{L}(X)$. Say that X is a *normed \mathbb{B} -space* if $\mathbb{B} \subset \mathcal{L}(X)$ and for every partition of unity $(b_\xi)_{\xi \in \Xi}$ in \mathbb{B} the two conditions hold:

- (1) If $b_\xi x = 0$ ($\xi \in \Xi$) for some $x \in X$ then $x = 0$;
- (2) If $b_\xi x = b_\xi x_\xi$ ($\xi \in \Xi$) for $x \in X$ and a family $(x_\xi)_{\xi \in \Xi}$ in X then

$$\|x\| \leq \sup\{\|b_\xi x_\xi\| : \xi \in \Xi\}.$$

Given a partition of unity (b_ξ) , we refer to $x \in X$ satisfying the condition $(\forall \xi \in \Xi) b_\xi x = b_\xi x_\xi$ as a *mixing* of (x_ξ) by (b_ξ) . If (1) holds then there is a unique mixing x of (x_ξ) by (b_ξ) . In these circumstances we naturally call x the *mixing* of (x_ξ) by (b_ξ) . Condition (2) maybe paraphrased as follows: The unit ball U_X of X is closed under mixing.

A normed \mathbb{B} -space X is *\mathbb{B} -cyclic* if we may find in X a mixing of each norm-bounded family by each partition of unity in \mathbb{B} . It is easy to verify that X is a \mathbb{B} -cyclic normed space if and only if, given a partition of unity $(b_\xi) \subset \mathbb{B}$ and a family $(x_\xi) \subset U_X$, we may find a unique element $x \in U_X$ such that $b_\xi x = b_\xi x_\xi$ for all ξ .

A linear operator (linear isometry) S between normed \mathbb{B} -spaces is *\mathbb{B} -linear* (*\mathbb{B} -isometry*) if S commutes with the projections in \mathbb{B} ; i.e., $\pi \circ S = S \circ \pi$ for all $\pi \in \mathbb{B}$. Denote by $\mathcal{L}_{\mathbb{B}}(X, Y)$ the set of all bounded \mathbb{B} -linear operators from X to Y . We call $X^\# := \mathcal{L}_{\mathbb{B}}(X, \mathbb{B}(\mathbb{R}))$ the *\mathbb{B} -dual* of X . If $X^\#$ and Y are \mathbb{B} -isometric to each other then we say that Y is a *\mathbb{B} -dual space* and X is a *\mathbb{B} -predual* of Y .

10.4. Theorem. *A Banach space X is linearly isometric to the bounded descent of some Banach space \mathcal{X} inside $\mathbb{V}^{(\mathbb{B})}$ (called a *Boolean valued representation* of X) if and only if X is \mathbb{B} -cyclic. If X and Y are \mathbb{B} -cyclic Banach spaces and \mathcal{X} and \mathcal{Y} stand for some Boolean valued representations of X and Y , then the space $\mathcal{L}_{\mathbb{B}}(X, Y)$ is \mathbb{B} -isometric to the bounded descent of the internal space $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ of all bounded linear operators from \mathcal{X} to \mathcal{Y} .*

10.5. Let Λ be a Stone algebra with unity $\mathbf{1}$ (= an order complete complex AM -space with strong order unity $\mathbf{1}$ and uniquely defined multiplicative structure) and consider a unitary Λ -module X . The mapping $\langle \cdot | \cdot \rangle : X \times X \rightarrow \Lambda$ is a *Λ -valued inner product*, if for all $x, y, z \in X$ and $a \in \Lambda$ the following are satisfied:

- (1) $\langle x | x \rangle \geq 0$; $\langle x | x \rangle = 0 \leftrightarrow x = 0$;
- (2) $\langle x | y \rangle = \langle y | x \rangle^*$;
- (3) $\langle ax | y \rangle = a \langle x | y \rangle$;

$$(4) \langle x + y | z \rangle = \langle x | z \rangle + \langle y | z \rangle.$$

Using a Λ -valued inner product, we may introduce the norm of $x \in X$ by

$$\|x\| := \sqrt{\|\langle x|x \rangle\|}$$

and the decomposable vector norm of $x \in X$ by

$$|x| := \sqrt{\langle x|x \rangle}.$$

Obviously, $\|x\| = \||x|\|$ for all $x \in X$, and so X is a space with mixed norm.

10.6. Let X be a Λ -module with an inner product $\langle \cdot | \cdot \rangle : X \times X \rightarrow \Lambda$. If X is complete with respect to the mixed norm $\|| \cdot \|$ then X is called a *C*-module* over Λ . It can be proved (see [33]) that for a C^* -module X the pair $(X, \|| \cdot \|)$ is a \mathbb{B} -cyclic Banach space if and only if $(X, | \cdot |)$ is a Banach–Kantorovich space over Λ . If a unitary C^* -module satisfies one of these equivalent conditions then it is called a *Kaplansky–Hilbert module*.

10.7. Theorem. *The bounded descent of a Hilbert space in $\mathbb{V}^{(\mathbb{B})}$ is a Kaplansky–Hilbert module over the Stone algebra $\mathcal{C}\Downarrow$. Conversely, if X is a Kaplansky–Hilbert module over $\mathcal{C}\Downarrow$, then there is a Hilbert space \mathcal{X} in $\mathbb{V}^{(\mathbb{B})}$ whose bounded descent is unitarily equivalent with X . This space is unique up to unitary equivalence inside $\mathbb{V}^{(\mathbb{B})}$.*

10.8. Theorem. *Let \mathcal{X} and \mathcal{Y} be Hilbert spaces inside $\mathbb{V}^{(\mathbb{B})}$. Suppose that X and Y are the bounded descents of \mathcal{X} and \mathcal{Y} . Then the space $\mathcal{L}_{\mathbb{B}}(X, Y)$ of all \mathbb{B} -linear bounded operators is a \mathbb{B} -cyclic Banach space \mathbb{B} -isometric to the bounded descent of the internal Banach space $\mathcal{L}^{\mathbb{B}}(\mathcal{X}, \mathcal{Y})$ of bounded linear operators from \mathcal{X} to \mathcal{Y} .*

10.9. Boolean valued analysis approach gives rise to an interesting concept of cyclically compact operator in a Banach \mathbb{B} -space [33, 8.5.5]. Without plunging into details we formulate a result on the general form of cyclically compact operators in Kaplansky–Hilbert modules.

Theorem. *Let X and Y be Kaplansky–Hilbert modules over a Stone algebra Λ and let T be a cyclically compact operator from X to Y . There are orthonormal families $(e_k)_{k \in \mathbb{N}}$ in X , $(f_k)_{k \in \mathbb{N}}$ in Y , and a family $(\mu_k)_{k \in \mathbb{N}}$ in Λ such that the following hold:*

- (1) $\mu_{k+1} \leq \mu_k$ ($k \in \mathbb{N}$) and $o\text{-}\lim_{k \rightarrow \infty} \mu_k = 0$;
- (2) there exists a projection π_{∞} in Λ such that $\pi_{\infty} \mu_k$ is a weak order unity in $\pi_{\infty} \Lambda$ for all $k \in \mathbb{N}$;
- (3) there exists a partition $(\pi_k)_{k=0}^{\infty}$ of the projection π_{∞}^{\perp} such that $\pi_0 \mu_1 = 0$, $\pi_k \leq \mu_k$, and $\pi_k \mu_{k+1} = 0$ for all $k \in \mathbb{N}$;
- (4) the representation is valid

$$\begin{aligned} T &= \pi_{\infty} \text{bo-} \sum_{k=1}^{\infty} \mu_k e_k^{\#} \otimes f_k \\ &+ \text{bo-} \sum_{n=1}^{\infty} \pi_n \sum_{k=1}^n \mu_k e_k^{\#} \otimes f_k. \end{aligned}$$

10.10. The bounded descent of 10.2 appeared in the research by G. Takeuti into von Neumann algebras and C^* -algebras within Boolean valued models [66, 67] and in the research by M. Ozawa into Boolean valued interpretation of the theory

of Hilbert spaces [49]. Theorems 10.4 and 10.9 were obtained by A. G. Kusraev in [30, 32, 33]. Theorems 10.7 and 10.8 were proved by M. Ozawa [49].

11. BANACH ALGEBRAS

The possibility of applying Boolean valued analysis to operator algebras rests on the following observation: If the center of an algebra is properly qualified and perfectly located then it becomes a one-dimensional subalgebra after ascending in a suitable Boolean valued universe. This might lead to a simpler algebra. On the other hand, the transfer principle implies that the scope of the formal theory of the initial algebra is the same as that of its Boolean valued representation.

11.1. An *AW*-algebra* is a C^* -algebra presenting a Baer $*$ -algebra. More explicitly, an *AW*-algebra* is a C^* -algebra A whose every *right annihilator* $M^\perp := \{y \in A : (\forall x \in M) xy = 0\}$ has the form pA , with p a projection. A *projection* p is a hermitian ($p^* = p$) idempotent ($p^2 = p$) element. An element $z \in A$ is said to be *central* if it commutes with every member of A . The *center* of an *AW*-algebra* A is the set $\mathcal{Z}(A)$ of all central elements. Clearly, $\mathcal{Z}(A)$ is a commutative *AW*-subalgebra* of A , with $\lambda \mathbb{1} \in \mathcal{Z}(A)$ for all $\lambda \in \mathbb{C}$. If $\mathcal{Z}(A) = \{\lambda \mathbb{1} : \lambda \in \mathbb{C}\}$ then the *AW*-algebra* A is called an *AW*-factor*.

The symbol $\mathfrak{P}(A)$ stands for the set of all projections of an involutive algebra A . Denote the set of all central projections by $\mathfrak{P}_c(A)$.

11.2. Theorem. Assume that \mathcal{A} is an *AW*-algebra* inside $\mathbb{V}^{(\mathbb{B})}$ and A is the bounded descent of \mathcal{A} . Then A is also an *AW*-algebra* and, moreover, $\mathfrak{P}_c(A)$ has an order-closed subalgebra isomorphic with \mathbb{B} . Conversely, let A be an *AW*-algebra* such that \mathbb{B} is an order-closed subalgebra of the Boolean algebra $\mathfrak{P}_c(A)$. Then there is an *AW*-algebra* \mathcal{A} in $\mathbb{V}^{(\mathbb{B})}$ whose bounded descent is $*$ - \mathbb{B} -isomorphic with A . This algebra \mathcal{A} is unique up to isomorphism inside $\mathbb{V}^{(\mathbb{B})}$.

Observe that if \mathcal{A} is an *AW*-factor* inside $\mathbb{V}^{(\mathbb{B})}$ then the bounded descent A of \mathcal{A} is an *AW*-algebra* whose Boolean algebra of central projections is isomorphic with \mathbb{B} . Conversely, if A is an *AW*-algebra* and $\mathbb{B} := \mathfrak{P}_c(A)$ then there is an *AW*-factor* \mathcal{A} inside $\mathbb{V}^{(\mathbb{B})}$ whose bounded descent is isomorphic with A .

11.3. Take an *AW*-algebra* A . Clearly, the formula

$$q \leq p \leftrightarrow q = qp = pq \quad (q, p \in \mathfrak{P}(X))$$

(sometimes reads as “ p contains q ”) specifies some order \leq on the set of projections $\mathfrak{P}(A)$. Moreover, $\mathfrak{P}(A)$ is a complete lattice and $\mathfrak{P}_c(A)$ is a complete Boolean algebra.

The classification of *AW*-algebras* into types is determined from the structure of its lattice of projections [33, 57]. It is important to emphasize that Boolean valued representation preserves this classification. We recall only the definition of type I *AW*-algebra*. A projection $\pi \in A$ is called *abelian* if the algebra $\pi A \pi$ is commutative. An algebra A has *type I*, if each nonzero projection in A contains a nonzero abelian projection.

We call an *AW*-algebra* *embeddable* if it is $*$ -isomorphic with the double commutant of some type I *AW*-algebra*. Each embeddable *AW*-algebra* admits a Boolean valued representation, becoming a von Neumann algebra or factor. A C^* -algebra A is called *\mathbb{B} -embeddable* if there is a type I *AW*-algebra* N and a $*$ -monomorphism

$\iota : A \rightarrow N$ such that $\mathbb{B} = \mathfrak{P}_c(N)$ and $\iota(A) = \iota(A)''$, where $\iota(A)''$ is the bicommutant of $\iota(A)$ in N . Note that in this event A is an AW^* -algebra and \mathbb{B} is a regular subalgebra of $\mathfrak{P}_c(A)$. In particular, A is a \mathbb{B} -cyclic algebra (see 10.3).

Say that a C^* -algebra A is *embeddable* if A is \mathbb{B} -embeddable for some regular subalgebra $\mathbb{B} \subset \mathfrak{P}_c(A)$. If $\mathbb{B} = \mathfrak{P}_c(A)$ and A is \mathbb{B} -embeddable then A is called a *centrally embeddable algebra*.

11.4. Theorem. *Let \mathcal{A} be a C^* -algebra inside $\mathbb{V}^{(\mathbb{B})}$ and let A be the bounded descent of \mathcal{A} . Then A is a \mathbb{B} -embeddable AW^* -algebra if and only if \mathcal{A} is a von Neumann algebra inside $\mathbb{V}^{(\mathbb{B})}$. The algebra A is centrally embeddable if and only if \mathcal{A} is a von Neumann factor inside $\mathbb{V}^{(\mathbb{B})}$.*

Using this representation, we can obtain characterizations of embeddable AW^* -algebras. In particular, an AW^* -algebra A is embeddable if and only if the center-valued normal states of A separate A .

11.5. Theorem. *For an AW^* -algebra A the following are equivalent:*

- (1) *A is embeddable;*
- (2) *A is centrally embeddable;*
- (3) *A has a separating set of center-valued normal states;*
- (4) *A is a $\mathfrak{P}_c(A)$ -predual space.*

11.6. Combining the results about the Boolean valued representations of AW^* -algebras with the analytical representations for dominated operators (see [33]), we come to some functional representations of AW^* -algebras.

Suppose that Q is an extremally disconnected compact space, H is a Hilbert space, and $B(H)$ is the space of bounded linear endomorphisms of H . Denote by $\mathfrak{C}(Q, B(H))$ the set of all operator-functions $u : \text{dom}(u) \rightarrow B(H)$ on the comeager sets $\text{dom}(u) \subset Q$ and continuous in the strong operator topology. Introduce some equivalence on $\mathfrak{C}(Q, B(H))$ by putting $u \sim v$ if and only if u and v agree on $\text{dom}(u) \cap \text{dom}(v)$.

If $u \in \mathfrak{C}(Q, B(H))$ and $h \in H$ then the vector-function $uh : q \mapsto u(q)h$ ($q \in \text{dom}(u)$) is continuous thus determining a unique element $\widetilde{uh} \in C_\infty(Q, H)$ from the condition $uh \in \widetilde{uh}$. If \tilde{u} is the coset of the operator-function $u : \text{dom}(u) \rightarrow B(H)$ then $\tilde{u}h := \widetilde{uh}$ ($h \in H$) by definition.

Denote by $SC_\infty(Q, B(H))$ the set of all cosets \tilde{u} such that $u \in \mathfrak{C}(Q, B(H))$ and the set $\{\|\tilde{u}h\| : \|h\| \leq 1\}$ is bounded in $C_\infty(Q)$. Put

$$\|\tilde{u}\| := \sup\{\|\tilde{u}h\| : \|h\| \leq 1\},$$

where the supremum is taken in $C_\infty(Q)$.

We naturally furnish $SC_\infty(Q, B(H))$ with the structure of a $*$ -algebra and unitary $C_\infty(Q)$ -module. We now introduce the following normed $*$ -algebra

$$SC_\#(Q, B(H)) := \{v \in SC_\infty(Q, B(H)) : \|v\| \in C(Q)\},$$

$$\|v\| = \|\|v\|\|_\infty \quad (v \in SC_\#(Q, B(H))).$$

11.7. Theorem. *To each type I AW^* -algebra A there exists a family of nonempty extremally disconnected compact spaces $(Q_\gamma)_{\gamma \in \Gamma}$ such that*

- (1) *Γ is a set of cardinals and Q_γ is γ -stable for every $\gamma \in \Gamma$;*
- (2) *there is a $*$ - \mathbb{B} -isomorphism:*

$$A \simeq \sum_{\gamma \in \Gamma}^{\oplus} SC_\#(Q_\gamma, B(l_2(\gamma))).$$

This family is unique up to congruence.

A cardinal number γ is *Q-stable* if γ^\wedge is a cardinal number inside $\mathbb{V}^{(\mathbb{B})}$ and Q is the Stone space of \mathbb{B} .

11.8. The study of C^* -algebras and von Neumann algebras by Boolean valued models was started by G. Takeuti with [66, 67]. Theorems 11.2, 11.4, and 11.5 were obtained by M. Ozawa [51, 52, 54]. Theorem 11.7 was established by A. G. Kusraev.

Boolean valued analysis of AW^* -algebras yields a negative solution to the I. Kaplansky problem of unique decomposition of a type I AW^* -algebra into the direct sum of homogeneous bands. M. Ozawa gave this solution in [52, 53]. The lack of uniqueness is tied with the effect of the cardinal shift that may happens on ascending into a Boolean valued model $\mathbb{V}^{(\mathbb{B})}$. The cardinal shift is impossible in the case when the Boolean algebra of central idempotents \mathbb{B} under study satisfies the countable chain condition, and so the decomposition in question is unique. I. Kaplansky established uniqueness of the decomposition on assuming that B satisfies the countable chain condition and conjectured that uniqueness fails in general [28].

12. JB-ALGEBRAS

We also consider similar problems for the so-called *JB*-algebras presenting some real nonassociative analogs of C^* -algebras.

12.1. Recall that a *JB*-algebra is a real Banach space A which is a unital Jordan algebra satisfying the conditions:

- (1) $\|xy\| \leq \|x\| \cdot \|y\| \quad (x, y \in A);$
- (2) $\|x^2\| = \|x\|^2 \quad (x \in A);$
- (3) $\|x^2\| \leq \|x^2 + y^2\| \quad (x, y \in A).$

The intersection of all maximal associative subalgebras of A is called the *center* of A and denoted by $\mathcal{Z}(A)$. Evidently $\mathcal{Z}(A)$ is an associative *JB*-algebra and every such algebra is isometrically isomorphic to the real Banach algebra $C(K)$ of continuous functions on a compact space K . If $\mathcal{Z}(A) = \mathbb{R} \cdot \mathbf{1}$ then A is said to be a *JB-factor*.

The idempotents of *JB*-algebras are also called *projections*. The set $\mathfrak{P}(A)$ of projections naturally underlies a complete lattice. The set $\mathfrak{P}_c(A)$ of all projections belonging to the center makes a Boolean algebra. Assume that \mathbb{B} is a subalgebra of $\mathfrak{P}_c(A)$. Then we say that A is a *\mathbb{B} -JB-algebra* if to each partition of unity $(e_\xi)_{\xi \in \Xi}$ in \mathbb{B} and each family $(x_\xi)_{\xi \in \Xi}$ in A there exists a unique *\mathbb{B} -mixing* $x := \text{mix}_{\xi \in \Xi} (e_\xi x_\xi)$; i.e., a unique $x \in A$ such that $e_\xi x_\xi = e_\xi x$ for all $\xi \in \Xi$. If $\mathbb{B}(\mathbb{R}) := \mathcal{R} \downarrow = \mathcal{Z}(A)$ then a *\mathbb{B} -JB-algebra* is also called a *centrally extended JB-algebra*.

Clearly, the unit ball of a *\mathbb{B} -JB-algebra* A is closed under *\mathbb{B} -mixing* and so A is a *\mathbb{B} -cyclic Banach space*. Therefore, from 9.4 we can arrive to

12.2. Theorem. *The bounded descent of a JB-algebra inside $\mathbb{V}^{(\mathbb{B})}$ is a \mathbb{B} -JB-algebra. Conversely, for every \mathbb{B} -JB-algebra A there exists a unique JB-algebra \mathcal{A} (up to isomorphism) whose bounded descent is isometrically \mathbb{B} -isomorphic to A . Moreover, $\llbracket \mathcal{A} \text{ is a JB-factor} \rrbracket = \mathbf{1}$ if and only if $\mathbb{B}(\mathbb{R}) = \mathcal{Z}(A)$.*

12.3. Let A be a *\mathbb{B} -JB-algebra* and $\Lambda := \mathbb{B}(\mathbb{R})$. An operator $\Phi \in A^\#$ is called a *Λ -valued state* if $\Phi \geq 0$ and $\Phi(\mathbf{1}) = \mathbf{1}$. A state Φ is *normal* if $\Phi(x) = o\text{-}\lim \Phi(x_\alpha)$ for every increasing net (x_α) in A with $x := \sup x_\alpha$.

If \mathcal{A} is a Boolean valued representation of A then the ascent $\phi := \Phi \uparrow$ of Φ is a normal state on \mathcal{A} . Conversely, if $\llbracket \phi \text{ is a normal state on } \mathcal{A} \rrbracket = \mathbf{1}$ then the

restriction to A of $\phi \downarrow$ is a Λ -valued normal state. We now give a characterization of the \mathbb{B} - JB -algebras that are \mathbb{B} -dual spaces.

12.4. Theorem. *For a \mathbb{B} - JB -algebra A the following are equivalent:*

- (1) *A is a \mathbb{B} -dual space;*
- (2) *A is monotone complete and admits a separating set of Λ -valued normal states.*

If one of these conditions holds then the part of $A_\#$ consisting of σ -continuous operators serves as a \mathbb{B} -predual of A .

This is just a Boolean valued interpretation of the following theorem by F. W. Shultz [60]: *a JB -algebra A is a dual Banach space if and only if A is monotone complete and has a separating family of normal states.*

12.5. An algebra A satisfying one of the equivalent conditions 12.4 (1,2), is called a \mathbb{B} - JBW -algebra. If, moreover, \mathbb{B} coincides with the set of all central projections then A is a \mathbb{B} - JBW -factor. It follows from Theorem 12.4 that A is a \mathbb{B} - JBW -algebra (\mathbb{B} - JBW -factor) if and only if its Boolean valued representation $\mathcal{A} \in \mathbb{V}^{(\mathbb{B})}$ is a JBW -algebra (JBW -factor).

Consider one example. Let X be a Kaplansky–Hilbert module over the algebra $\overline{\Lambda} := \mathbb{B}(\mathbb{C}) := \mathcal{C} \downarrow$. Then X is a \mathbb{B} -cyclic Banach space and $\mathcal{L}_{\mathbb{B}}(X)$ is a type I AW^* -algebra.

Given $x, y \in X$, define the seminorm

$$p_{x,y}(a) := \|\langle ax, y \rangle\|_\infty \quad (a \in \mathcal{L}_{\mathbb{B}}(X)),$$

where $\langle \cdot, \cdot \rangle$ is the inner product on X with values in $\overline{\Lambda}$. Denote by σ_∞ the topology on $\mathcal{L}_{\mathbb{B}}(X)$ that is generated by the system of seminorms $p_{x,y}$. Then a σ_∞ -closed \mathbb{B} - JB -algebra of selfadjoint operators presents an example of a \mathbb{B} - JBW -algebra.

12.6. Let A be an associative algebra over a field of characteristic $\neq 2$. Define the new multiplication $a \circ b := 1/2(ab + ba)$ on the vector space of A . Denote the resulting algebra by A^J . This A^J is a Jordan algebra. If the subspace A_\circ of A is closed under $a \circ b$ then A_\circ is a subalgebra of A^J and so A_\circ is Jordan. Such a Jordan algebra A_\circ is called *special*. The nonspecial Jordan algebras are referred to as *exceptional*.

Let \mathbb{O} be the *Cayley* or *octonian* algebra. Let $M_n(\mathbb{O})$ be the algebra of $n \times n$ -matrices with entries in \mathbb{O} . The involution $*$ on $M_n(\mathbb{O})$ is as usual the transposition of a matrix followed by conjugation of every entry. The set $M_n(\mathbb{O})_{\text{sa}} := \{x \in M_n(\mathbb{O}) : x^* = x\}$ of hermitian matrices is closed in $M_n(\mathbb{O})$ under the Jordan multiplication $x \circ y = 1/2(xy + yx)$. The real vector space $M_n(\mathbb{O})_{\text{sa}}$ is a Jordan algebra under \circ only for $n \leq 3$. In case $n = 1, 2$ we arrive at special Jordan algebras. The Jordan algebra $M_3(\mathbb{O})_{\text{sa}}$ is special and denoted by M_3^8 .

12.7. Theorem. *A special \mathbb{B} - JB -algebra A is a \mathbb{B} - JBW -algebra if and only if A is isomorphic to a σ_∞ -closed \mathbb{B} - JB -subalgebra of $\mathcal{L}_{\mathbb{B}}(X)_{\text{sa}}$ for some Kaplansky–Hilbert module X .*

12.8. Theorem. *Each \mathbb{B} - JBW -factor A admits a unique decomposition $A = eA \oplus e^*A$ with a central projection $e \in \mathbb{B}$, $e^* := 1 - e$, such that the algebra eA is special and the algebra e^*A is purely exceptional. Moreover, eA is \mathbb{B} -isomorphic to a σ_∞ -closed subalgebra of selfadjoint endomorphisms of some AW^* -module and e^*A is isomorphic to $C(Q, M_3^8)$, where Q is the Stone compact space of the Boolean algebra $e^*\mathbb{B} := [0, e^*]$.*

12.9. The JB -algebras are nonassociative real analogs of C^* -algebras and von Neumann operator algebras. The theory of these algebras stems from the article of P. Jordan, J. von Neumann, and E. Wigner [20] and exists as a branch of functional analysis since the mid 1960s, when D. M. Topping [68] and E. Størmer [62] have started the study of the nonassociative real analogs of von Neumann algebras, the JW -algebras presenting weakly closed Jordan algebras of bounded selfadjoint operators in a Hilbert space. The steps of development are reflected in [9, 6, 18]. The Boolean valued approach to JB -algebras is outlined by A. G. Kusraev. More details and references are collected in [37].

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